Z-Transform

ECE 3640 Discrete-Time Signals and Systems
Utah State University
Spring 2020

What is the z-transform?

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad z \in \mathrm{ROC}$$

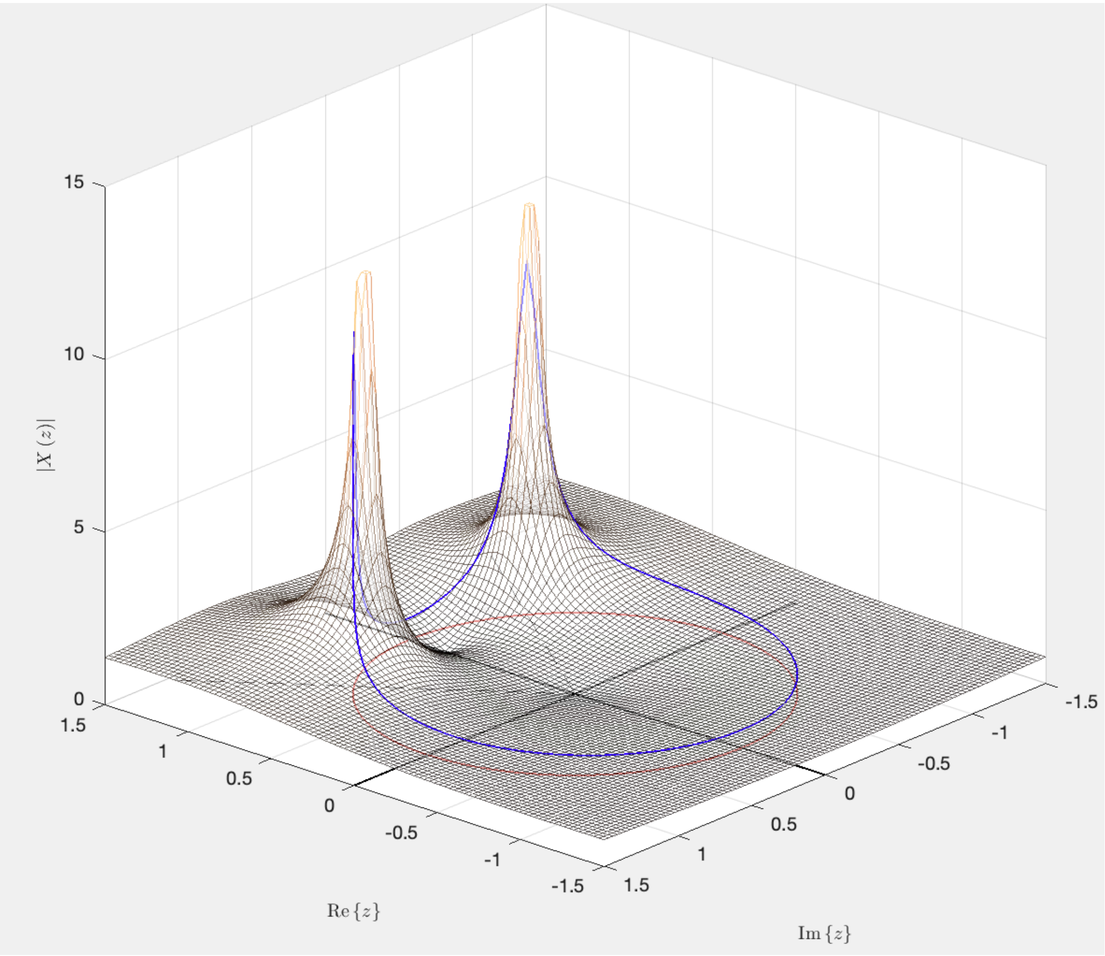
$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz, \quad C \in \mathrm{ROC}$$

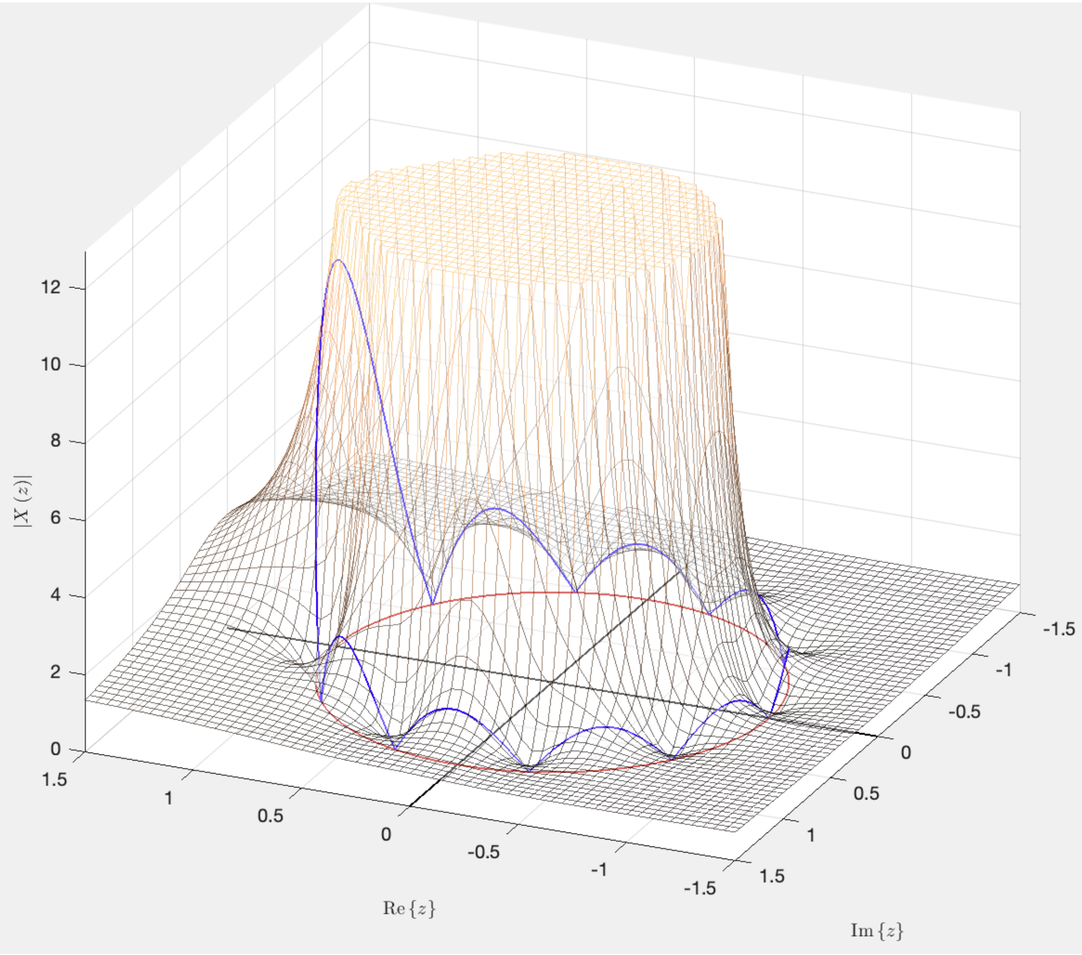
The z-transform is a function $X(z)$
along with a region of convergence (ROC).

The ROC is the set of points in the z-plane where $X(z)$ is finite, i.e. where $\sum_{n=-\infty}^{\infty} x[n] z^{-n}$ converges.

What is the z-transform?

It’s a function defined over the z-plane.





(Plots attributed to Oktay Alkin)

The mesh is $|X(z)|$.

The red line is the unit circle $\{z : |z| = 1\} = \{z : z = e^{j2\pi f}\}$.

The blue line is $X(z)|_{z=e^{j2\pi f}} = X(e^{j2\pi f})$.

How is the z-transform related to the DTFT?

Substitute $z = re^{j2\pi f}$ into z-transform definition.

$$X(z) = X(re^{j2\pi f})$$

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j2\pi fn} = \mathrm{DTFT}\{x[n]r^{-n}\}$$

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$$\boxed{X(f) = X(z)\big|_{z=e^{j2\pi f}}}$$

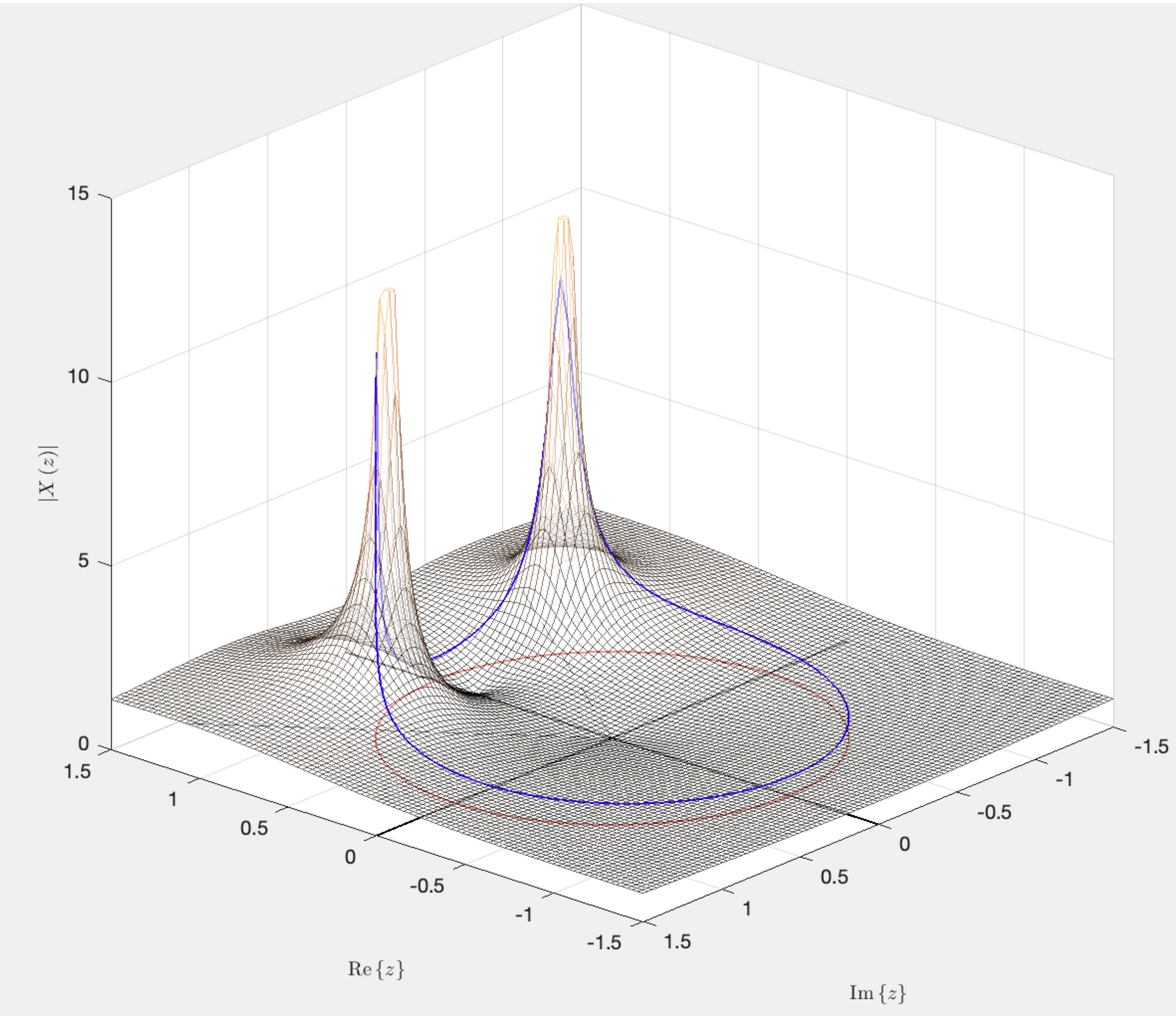
unit circle

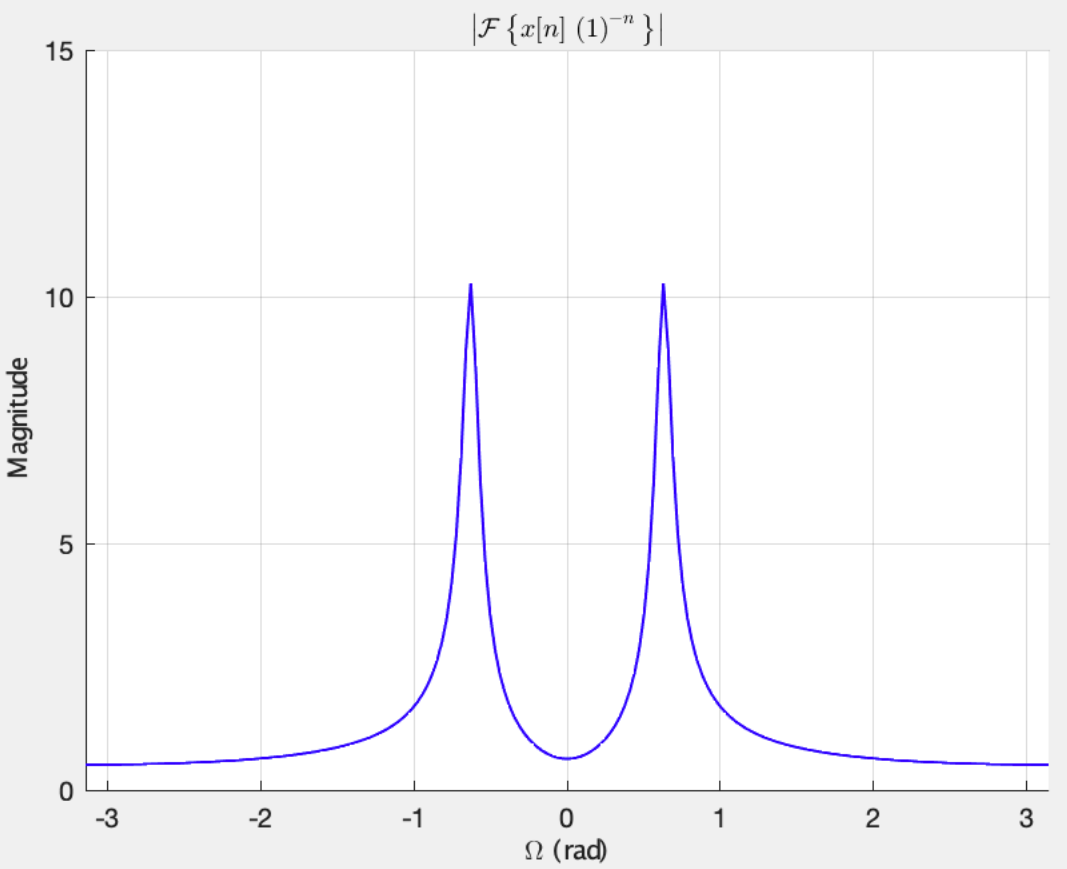
The DTFT is the z-transform evaluated on the unit circle.

How is the z-transform related to the DTFT?

$$X(f) = X(z)\big|_{z=e^{j2\pi f}}$$

The DTFT is the z-transform evaluated on the unit circle.



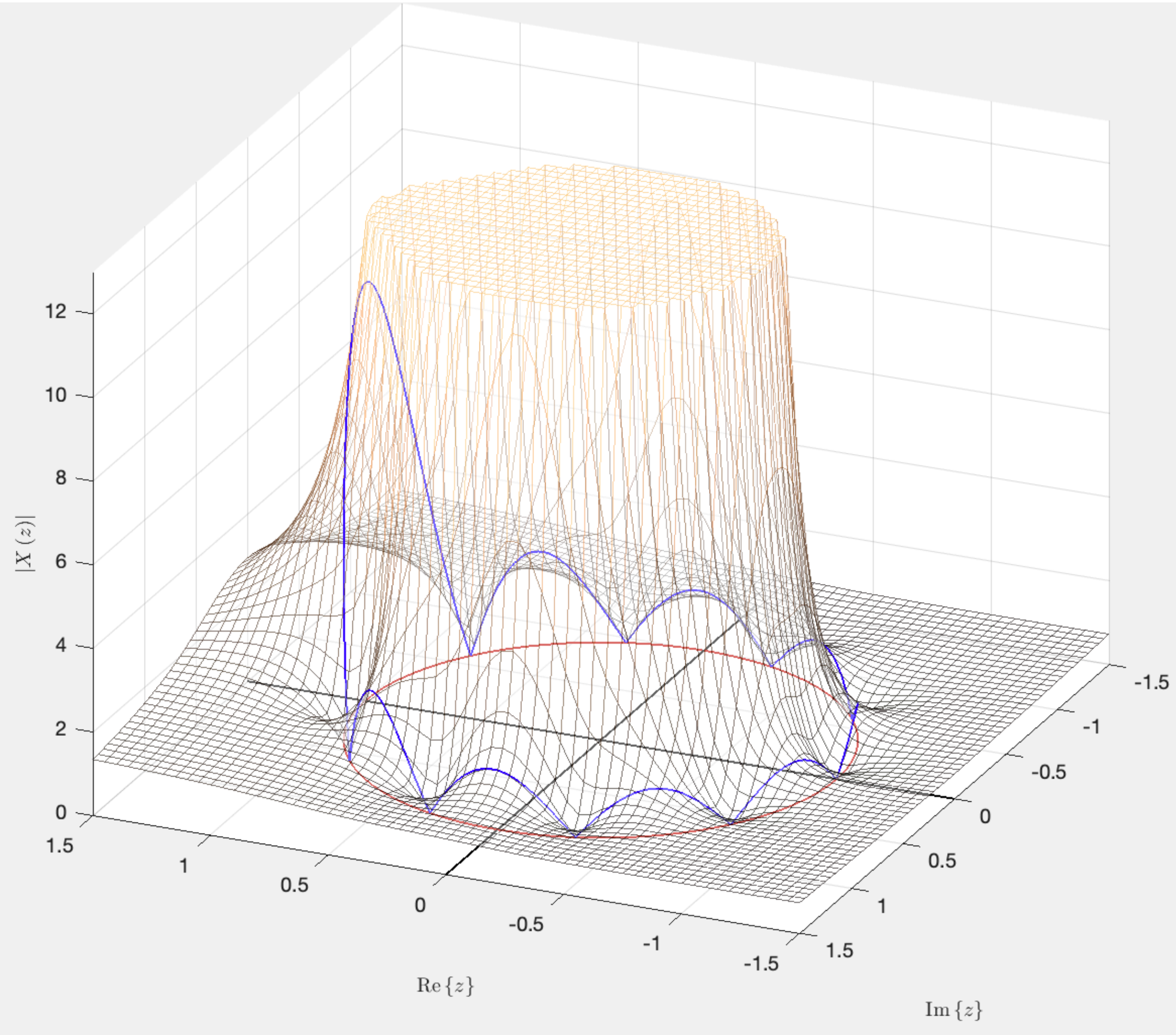


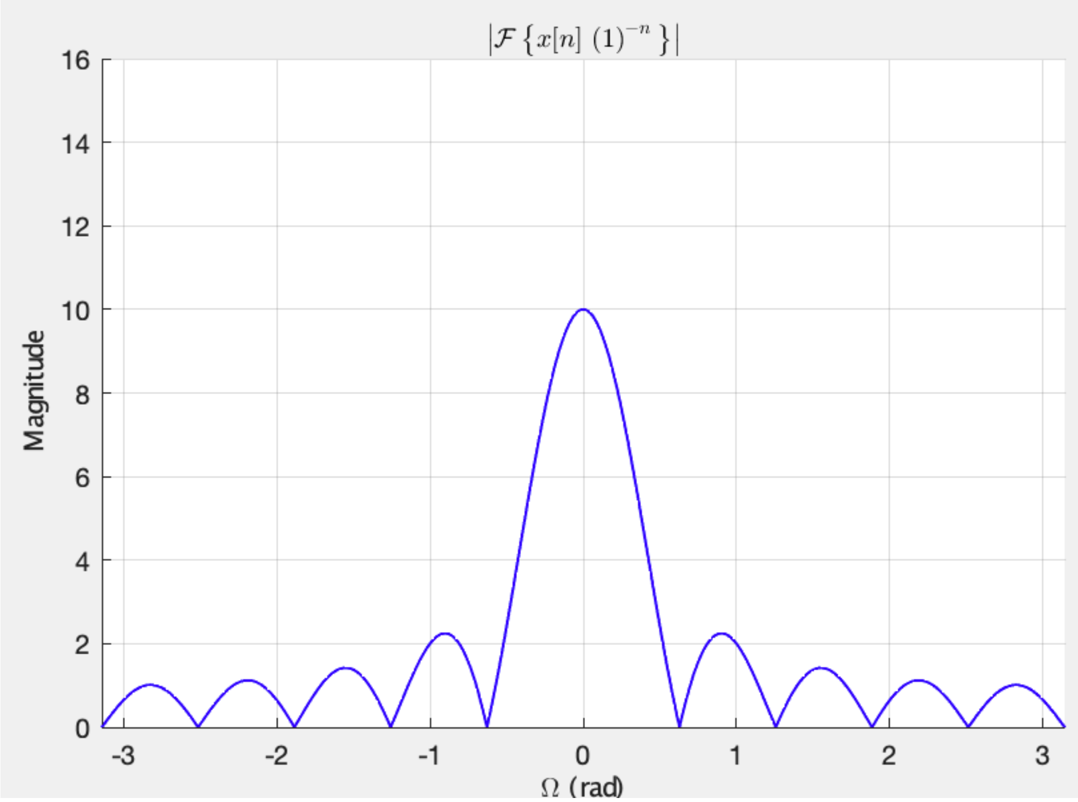
(Plots attributed to Oktay Alkin)

How is the z-transform related to the DTFT?

$$\boxed{X(f) = X(z)\big|_{z=e^{j2\pi f}}}$$

The DTFT is the z-transform evaluated on the unit circle.





(Plots attributed to Oktay Alkin)

Example

$$x[n] = a^n u[n], \quad |a| < 1$$

DTFT

$$X(f) = \sum_{n=0}^{\infty} a^n e^{-j2\pi f n} = \frac{1}{1 - ae^{-j2\pi f}}$$

z-Transform

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |az^{-1}| < 1$$

ROC

$$\boxed{X(f) = X(z)\big|_{z=e^{j2\pi f}}}$$

z-Transform, DTFT, DFT Relations

Let $x[n]$ be a length L signal and let $N \geq L$.
Now consider three transformations.

ZT $$X(z) = \sum_{n=0}^{L-1} x[n] z^{-n}$$

DTFT $$X(f) = \sum_{n=0}^{L-1} x[n] e^{-j2\pi f n} = X(z)|_{z=e^{j2\pi f}}$$

DFT $$X[k] = \sum_{n=0}^{L-1} x[n] e^{-j\frac{2\pi k n}{N}} = X(f)|_{f=\frac{k}{N}} = X(z)|_{e^{j\frac{2\pi k}{N}}}$$

$$\boxed{X(f) = X(z)|_{z=e^{j2\pi f}}}$$

$$\boxed{X[k] = X(z)|_{z=e^{j\frac{2\pi k}{N}}}}$$

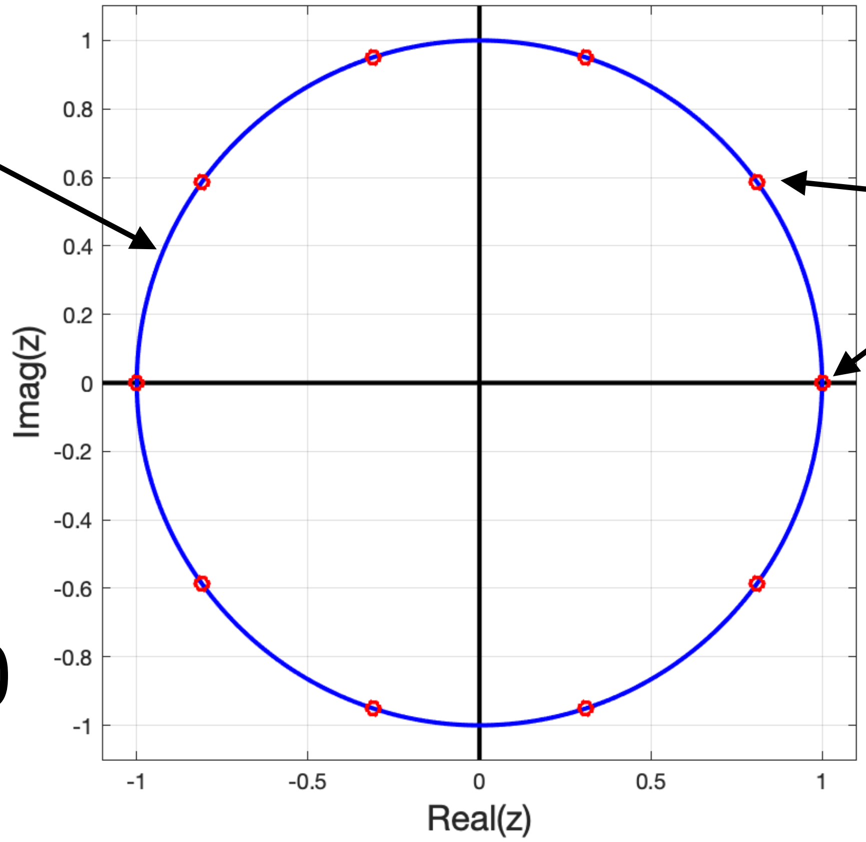
z-Transform, DTFT, DFT Relations

The DTFT is the z-transform evaluated on the unit circle.

$$X(f) = X(z)\big|_{z=e^{j2\pi f}}$$

$$X[k] = X(z)\Big|_{z=e^{j\frac{2\pi k}{N}}}$$

The DFT is the z-transform evaluated at N uniformly spaced points around the unit circle.



unit circle (blue)

uniformly spaced points (red)

Example: $N = 10$

Convergence and Regions of Convergence

DTFT

- Absolutely summable: $\sum_n |x[n]| < \infty \quad \Rightarrow \quad X(f)$ converges uniformly to a continuous (differentiable) function

- Energy signals: $\sum_n |x[n]|^2 < \infty \quad \Rightarrow \quad X(f)$ converges in the mean-square and has discontinuities

- Power signals: $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty \quad \Rightarrow \quad X(f)$ does not converge at some frequencies (Dirac delta functions)

DFT

- Finite length signals ⇒ always converges

Z-Transform

- Converges for all $z = re^{j2\pi f}$ where $x[n]r^{-n}$ is absolutely summable.

- Signals that do not have z-transforms: $e^{j2\pi f_0 n}$, periodic signals, constants, a^n, etc.

Convergence and Regions of Convergence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j2\pi fn}$$

The set of points where this sum converges is the region of convergence (ROC).

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j2\pi fn} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| \underbrace{\boxed{\left|e^{-j2\pi fn}\right|}}_{=1}$$

$$= \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

Questions about convergence for a given z depend only on the radius but not the angle.

This means that ROCs are circular.

Convergence and Regions of Convergence

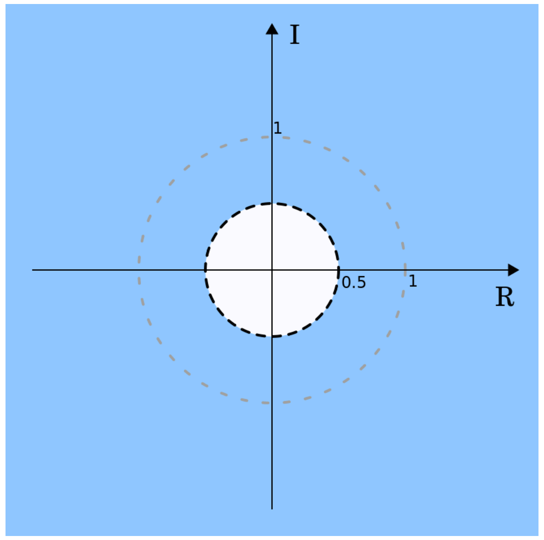
The ROC is a connected region.

The ROC may not contain poles.

In general, the ROC is either outside a circle, inside a circle, or in an annular region.

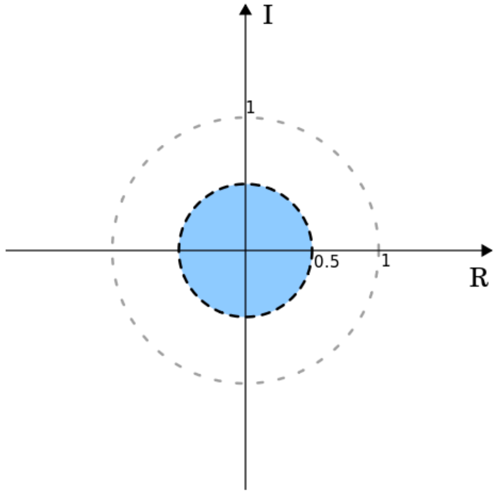
$|z| > |a|$

outside a circle



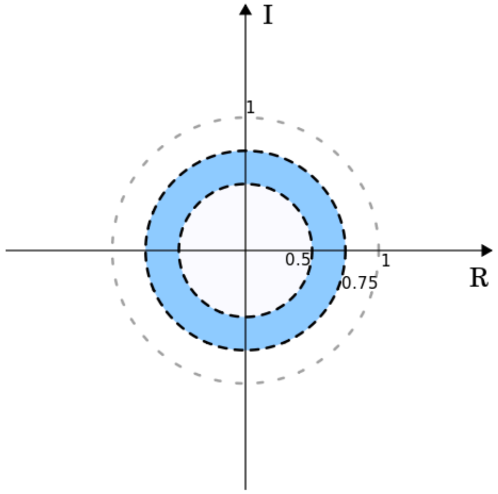
$|z| < |b|$

inside a circle



$|a| < |z| < |b|$

in an annular region

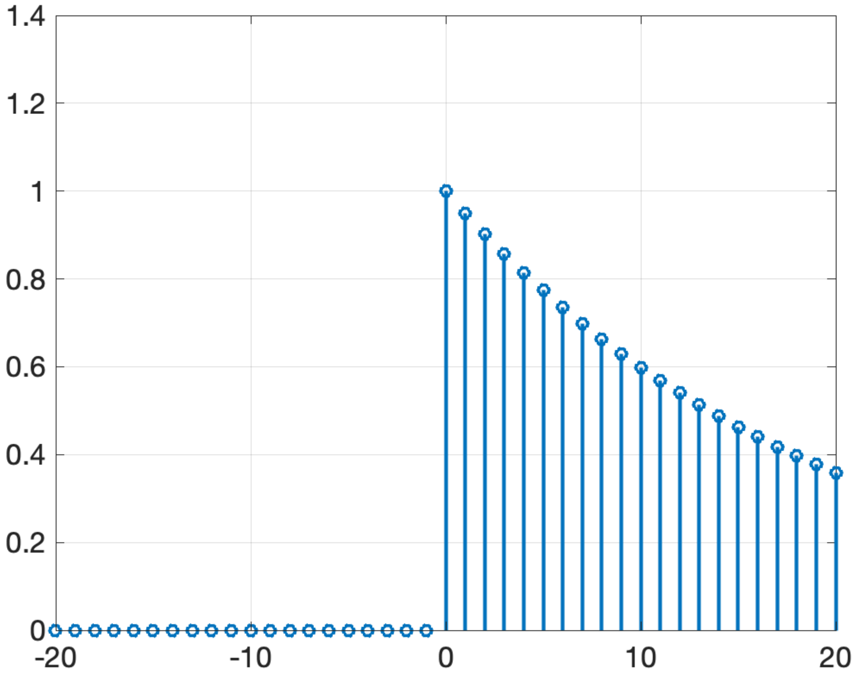


(pictures from Wikipedia)

Z-Transforms and ROCs by Example

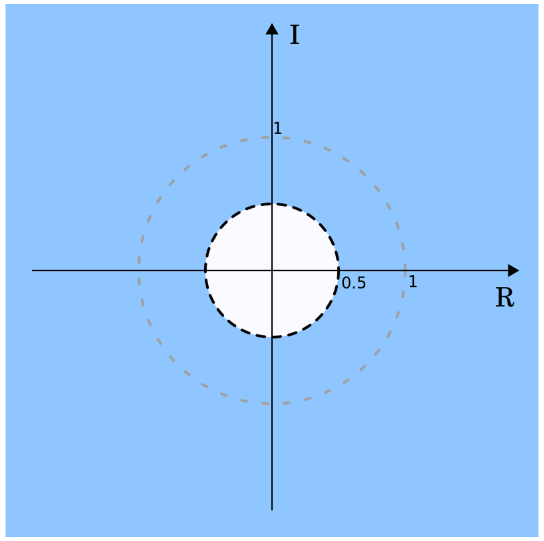
$x[n] = a^n u[n]$ (causal)

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, |az^{-1}| < 1 \quad \Leftrightarrow \quad |a| < |z|$$



$|z| > |a|$

outside a circle



Z-Transforms and ROCs by Example

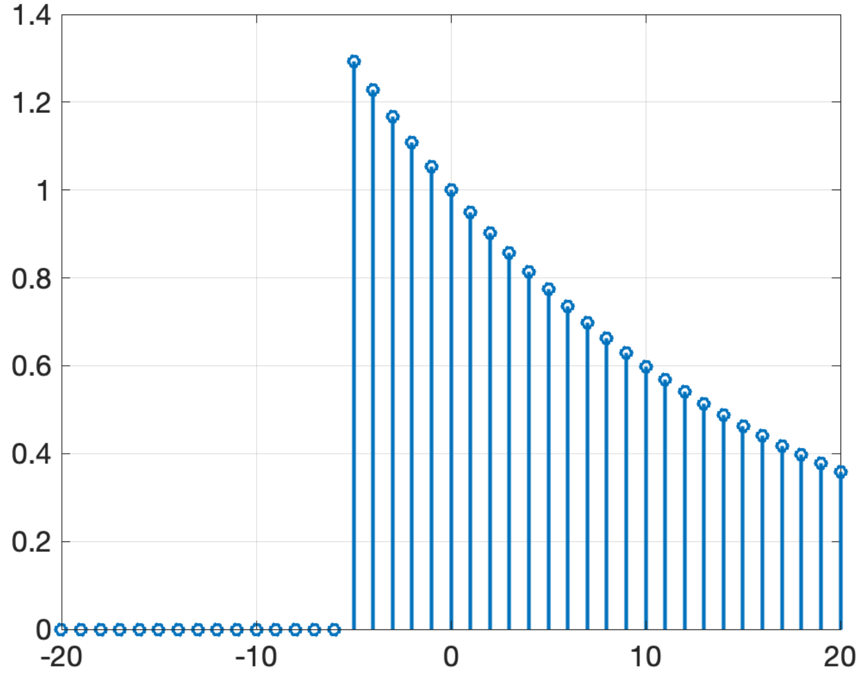
$x[n] = a^n u[n+5]$ (non-causal right-sided)

$$X(z) = \sum_{n=-5}^{\infty} a^n z^{-n} = \frac{a^{-5} z^5}{1 - az^{-1}}, |az^{-1}| < 1 \quad \Leftrightarrow \quad |a| < |z|$$

Notice that $z = \infty$ is also excluded

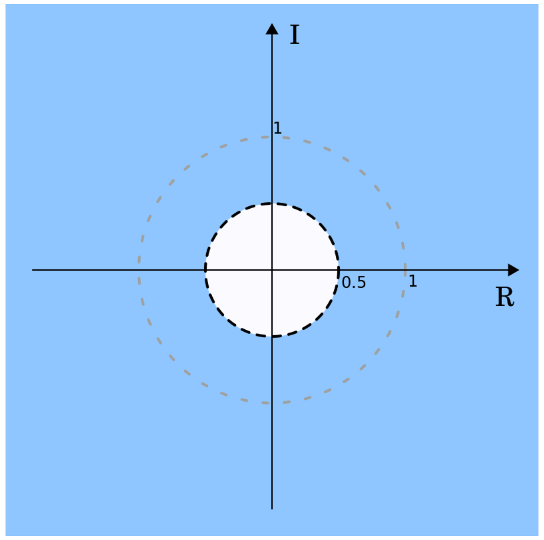
because it makes the numerator infinite.

So the ROC is $|a| < |z| < \infty$.



$|a| < |z| < \infty$

outside a circle

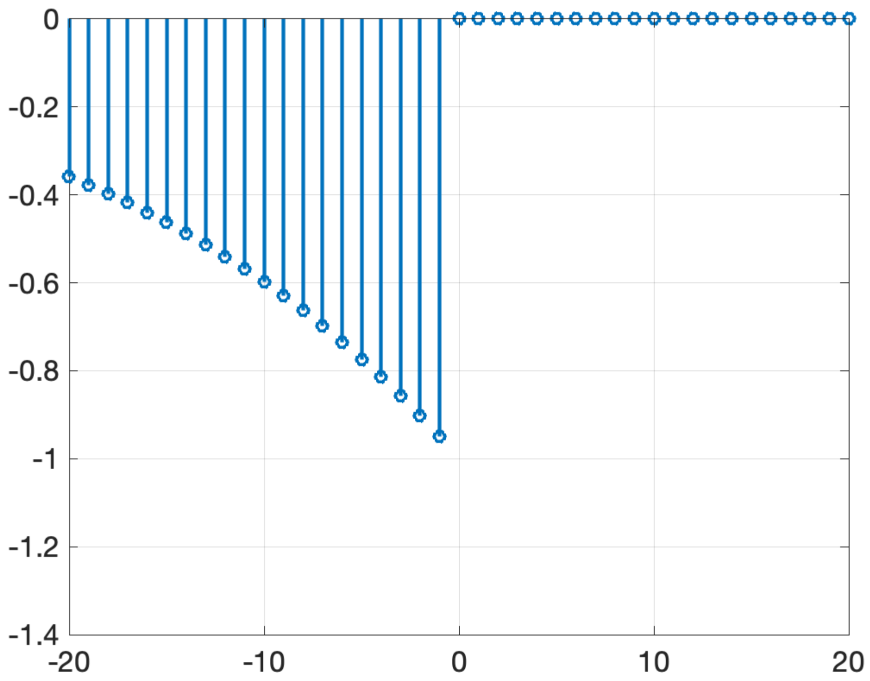


Z-Transforms and ROCs by Example

$$x[n] = -a^n u[-n-1]$$ (anti-causal)

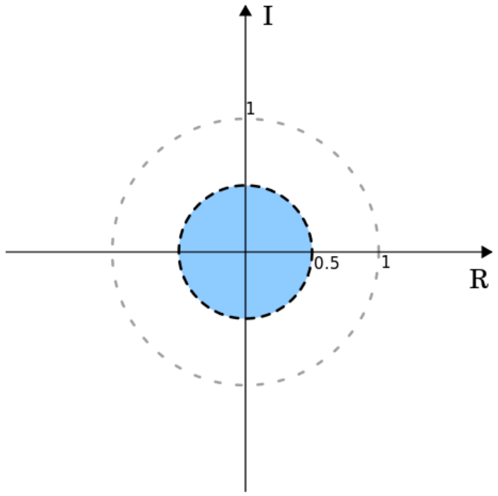
$$X(z) = \sum_{n=-\infty}^{-1} a^n z^{-n} = \sum_{m=1}^{\infty} a^{-m} z^m$$

$$= \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}}, |a^{-1}z| < 1 \quad \Leftrightarrow \quad |z| < |a|$$



$|z| < |a|$

inside a circle



Z-Transforms and ROCs by Example

$x[n] = -a^n u[-n+4]$ (nonanti-causal left-sided)

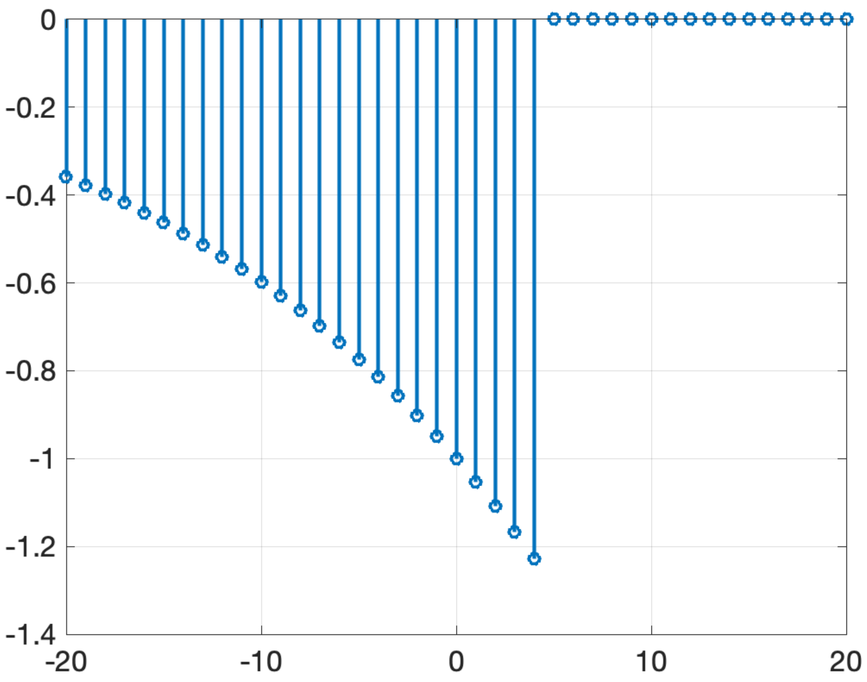
$$X(z) = -\sum_{n=-\infty}^{4} a^n z^{-n} = -\sum_{m=-4}^{\infty} a^{-m} z^m$$

$$= \frac{-a^4 z^{-4}}{1 - a^{-1} z} = \frac{a^5 z^{-5}}{1 - a z^{-1}}, |a^{-1} z| < 1 \quad \Leftrightarrow \quad |z| < |a|$$

Notice that $z = 0$ is also excluded

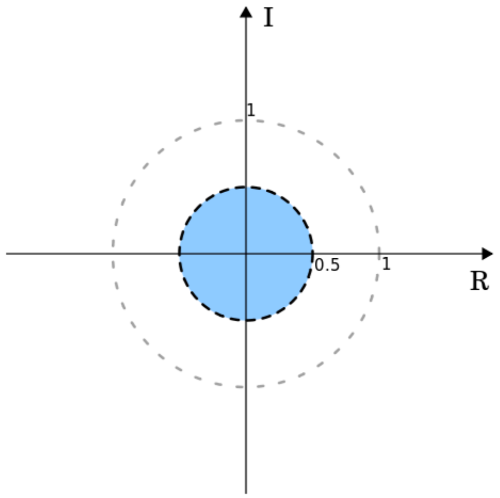
because it makes the numerator infinite.

So the ROC is $0 < |z| < |a|$.



$0 < |z| < |a|$

inside a circle



Convergence and Regions of Convergence

$$x[n] = a^n u[n] - b^n u[-n-1]$$ (doubly infinite)

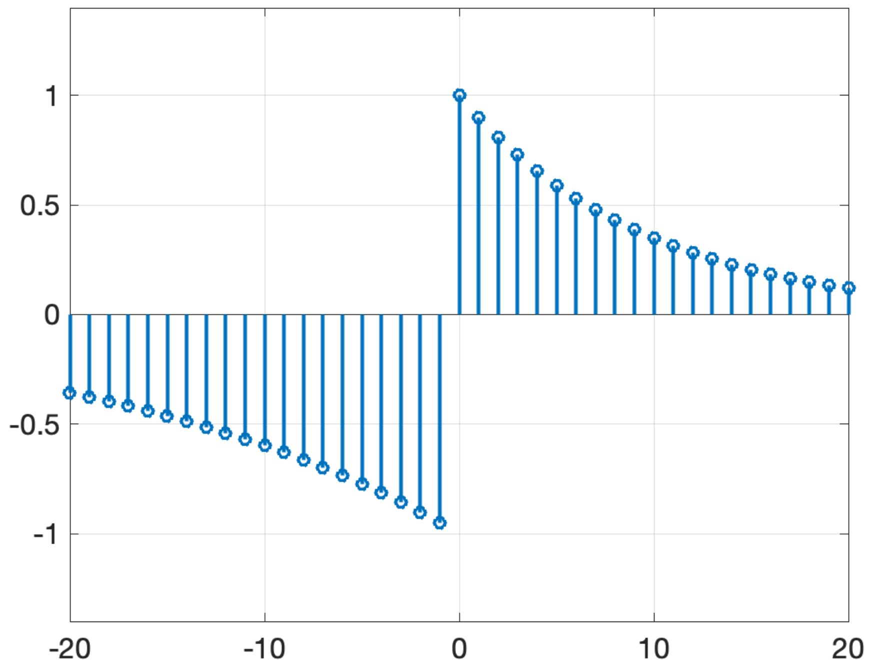
$$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}$$

$|z| > |a|$

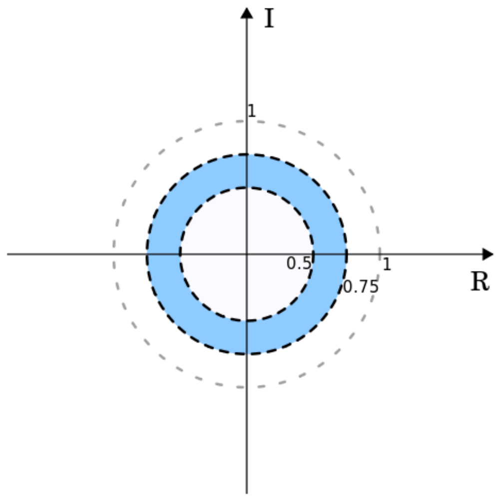
$|z| < |b|$

The combined ROC is $|a| < |z| < |b|$.

Requires that $|a| < |b|$.

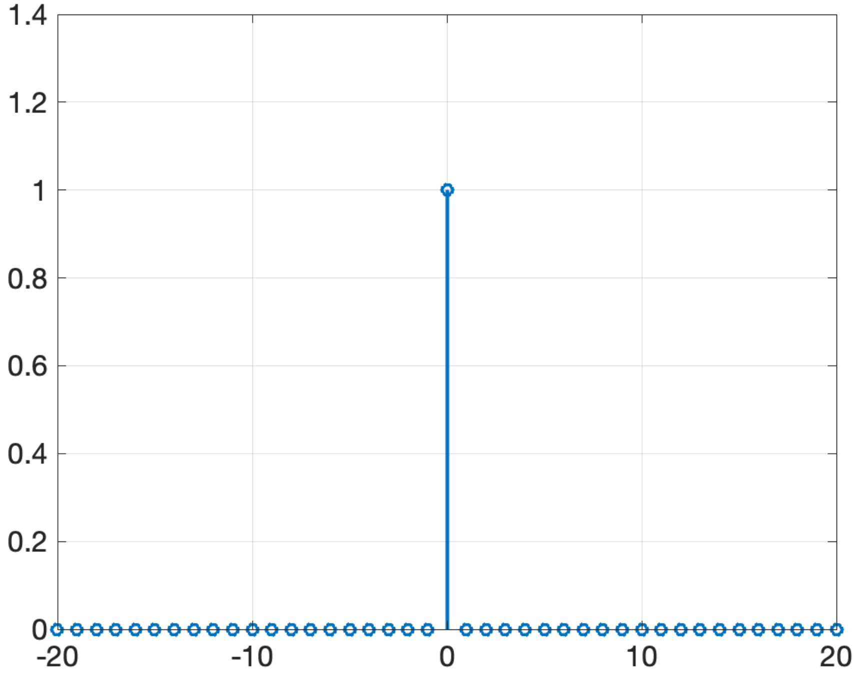


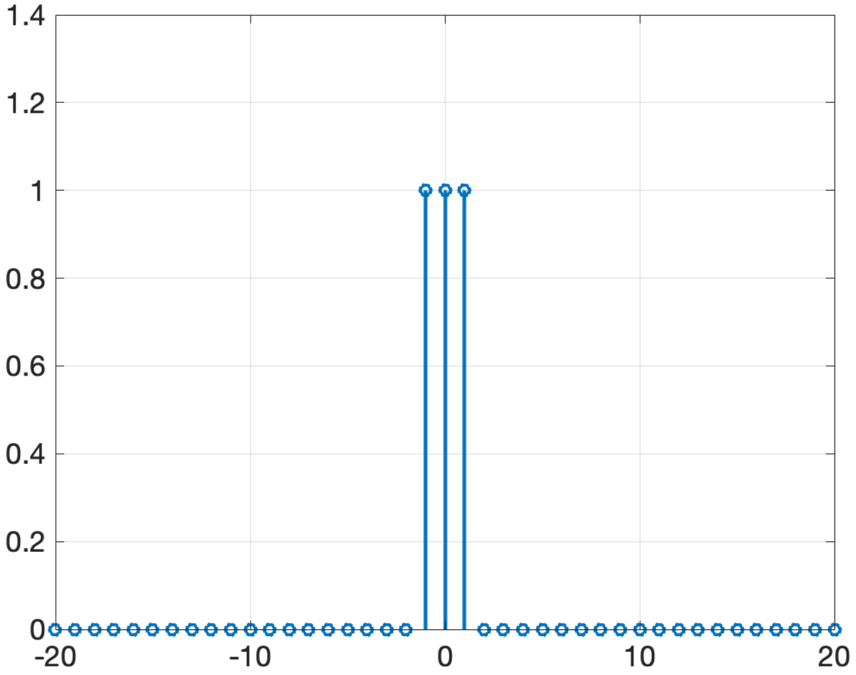
in an annular region



Convergence and Regions of Convergence

$$x[n] = \delta[n] \quad \longleftrightarrow \quad X(z) = 1, \quad \text{ROC} = \text{entire } z\text{-plane}$$





$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

$$X(z) = z + 1 + z^{-1}, \quad 0 < |z| < \infty$$

Convergence and Regions of Convergence

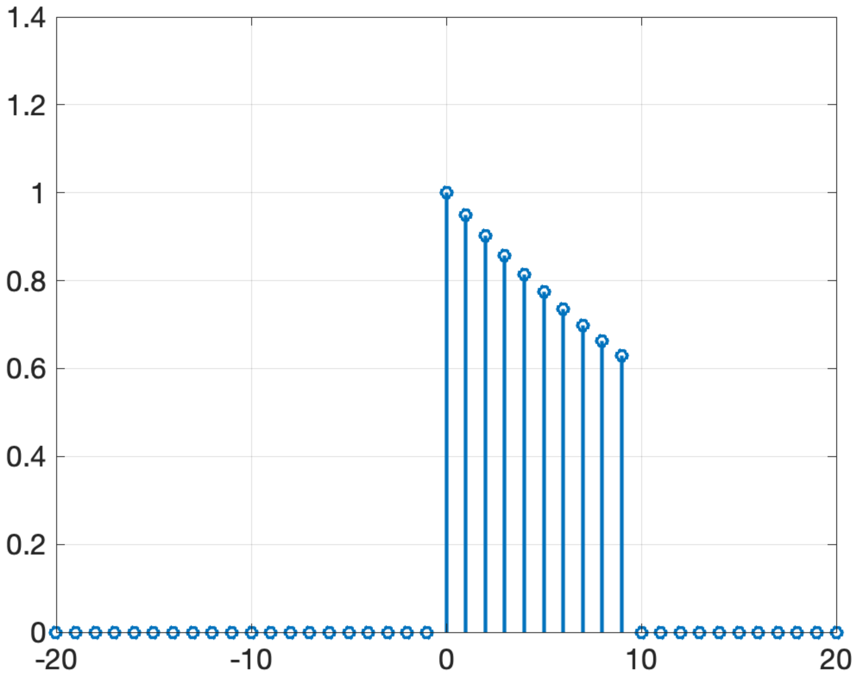
$$x[n] = a^n(u[n] - u[n-M])$$ (causal & finite length)

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} = 1 + az^{-1} + a^2z^{-2} + \cdots, a^{N-1}z^{-(N-1)}$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}}, \quad |z| > 0$$

Cannot put $z = 0$ into these terms.

This is the entire z-plane except the origin.



Convergence and Regions of Convergence

$$x[n] = a^n(u[n+N] - u[n-M])$$ (noncausal & finite length)

$$X(z) = \sum_{n=-N}^{M-1} a^n z^{-n} = a^{-N} z^{N} + \cdots + 1 + \cdots , a^{M-1} z^{-(M-1)}$$

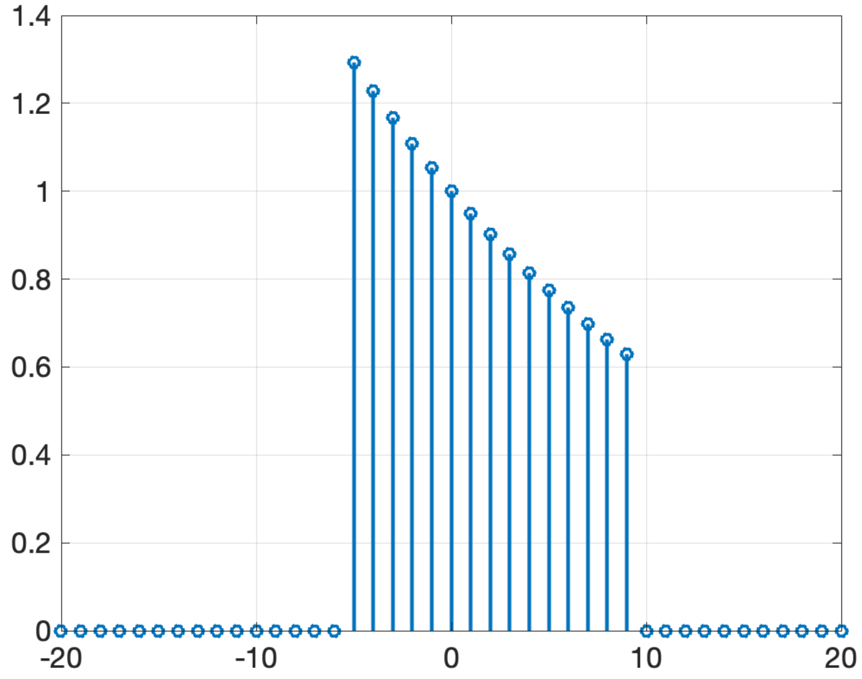
$|z| < \infty$

$|z| > 0$

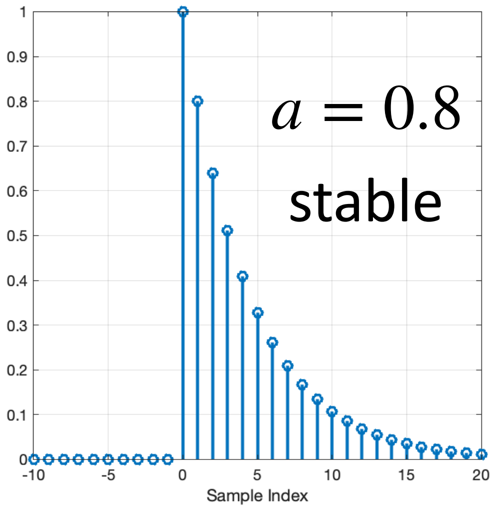
$$= \frac{(az^{-1})^N - (az^{-1})^M}{1 - az^{-1}},$$

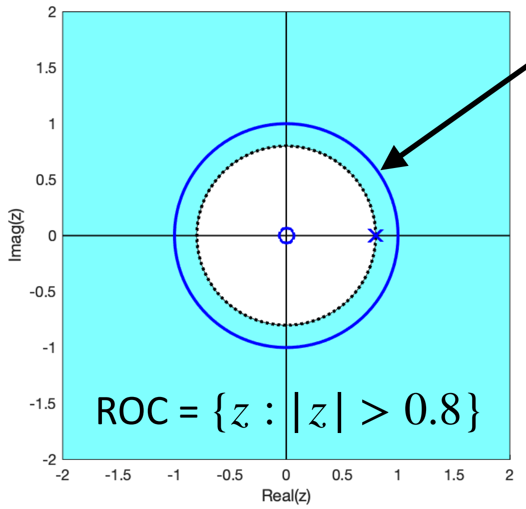
$$0 < |z| < \infty$$

This is the entire z-plane except $z = 0$ and $z = \infty$.



What can the ROC tell us? Stability & DTFT existence





unit circle (UC) = $\{z : |z| = 1\}$

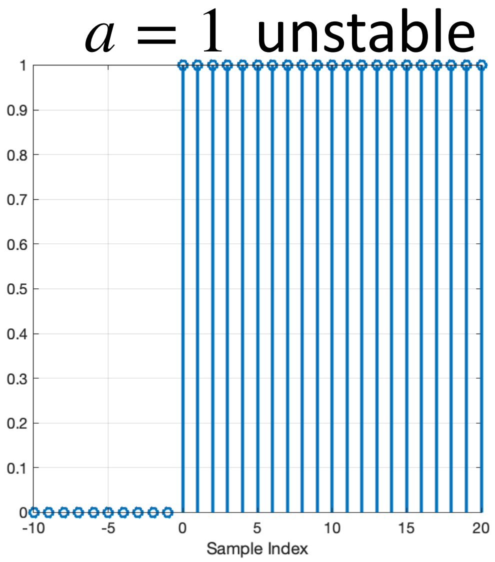
$$\boxed{X(f) = X(z)|_{z=e^{j2\pi f}}}$$

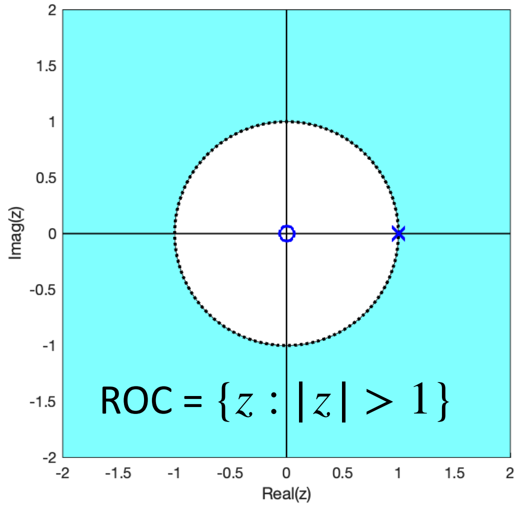
For DTFT to exist, ROC must include UC.

UC $\subset$ ROC $\Rightarrow$ DTFT exists

Stable $\Leftrightarrow$ Summable $\Leftrightarrow$ DTFT exists

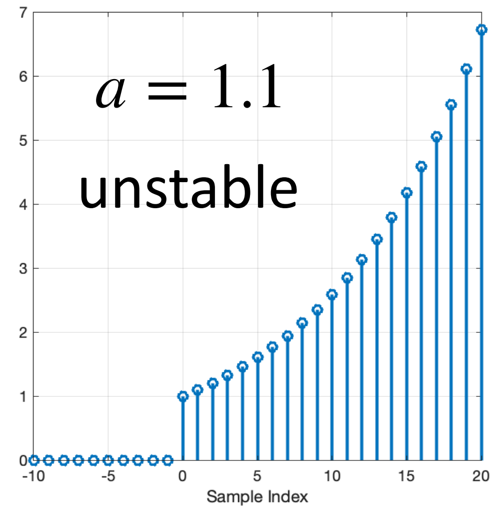
$$\sum_n |x[n]| < \infty$$

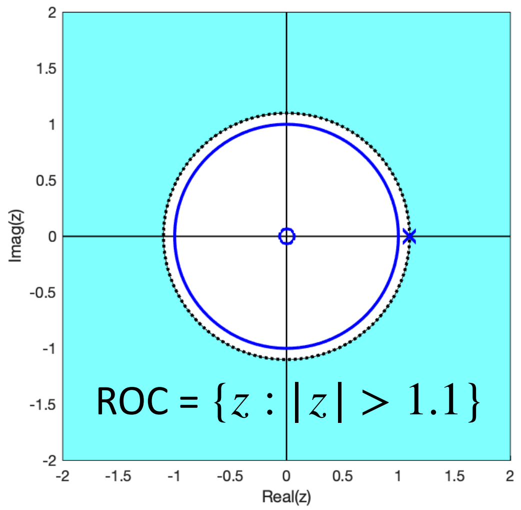




UC $\not\subset$ ROC $\Rightarrow$ DTFT does not exist

Unstable sequences have z-transforms but not DTFTs.





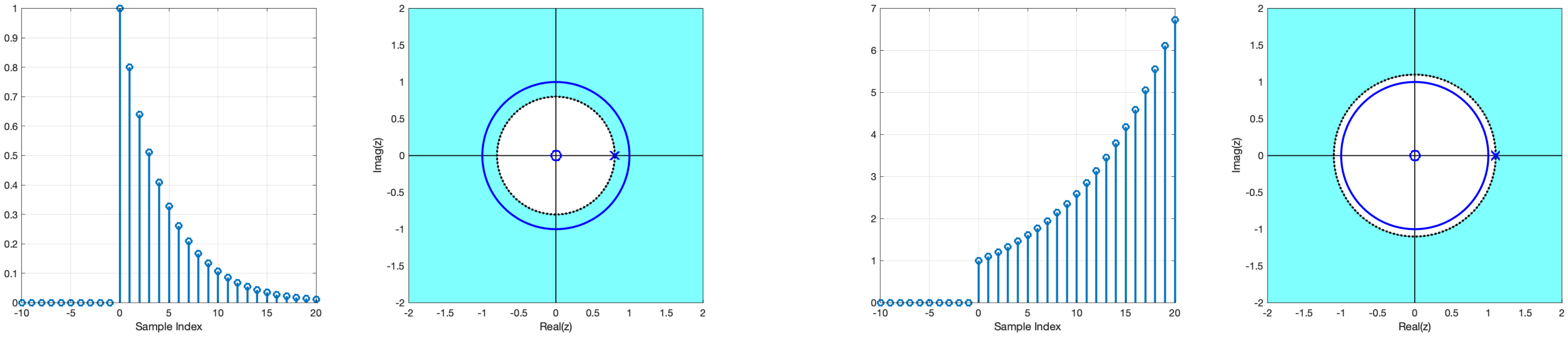
UC $\not\subset$ ROC $\Rightarrow$ DTFT does not exist

How can an unstable signal have a z-transform?

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n r^{-n} e^{-j2\pi f n} = \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n e^{-j2\pi f n}$$

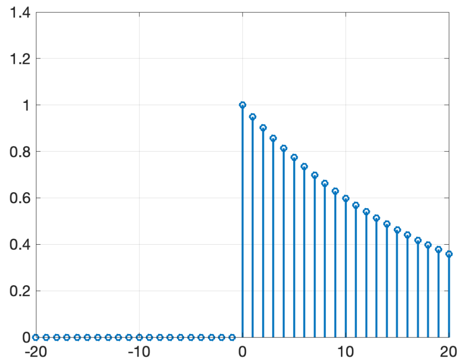
$z = re^{j2\pi f}$

Even if $a > 1$ so that a^n "blows up", r may be chosen so that $\left(\frac{a}{r}\right)^n$ decays. This requires that $r > |a|$ and explains why the ROC is $\{z : |z| = r > |a|\}$.



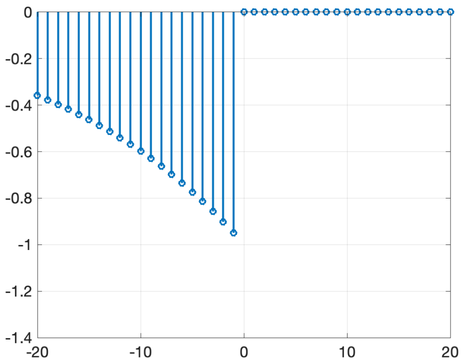
Why the ROC is so important

causal



These two sequences have the same $X(z)$ but different ROCs.

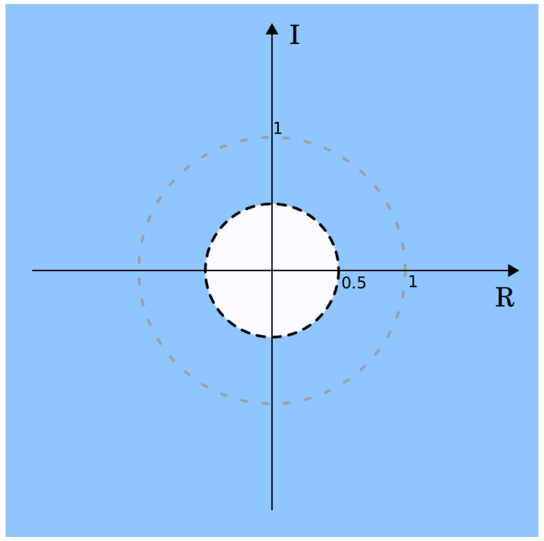
anti-causal



$$x[n] = a^n u[n]$$

$$|z| > |a|$$

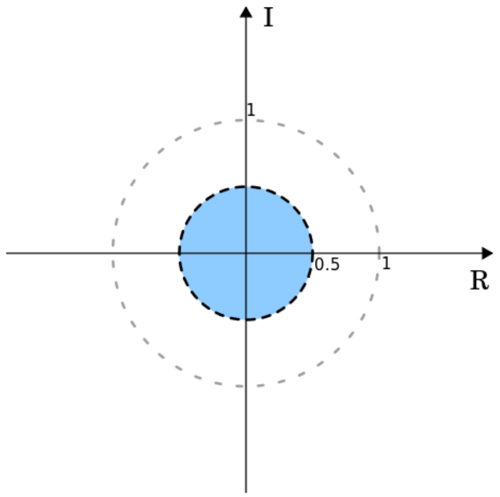
outside a circle



$$x[n] = -a^n u[-n-1]$$

$$|z| < |a|$$

inside a circle



$$\boxed{X(z) = \frac{1}{1 - az^{-1}}}$$

Always specify $X(z)$ and ROC.

Region of convergence

Q: Compute the z-transform for the sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1].$$

A: This is easy because we already know the transforms of the two parts:

$$\begin{aligned} \left(\frac{1}{2}\right)^n u[n] \quad &\rightarrow \quad \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \\ -\left(-\frac{1}{3}\right)^n u[-n-1] \quad &\rightarrow \quad \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| < \frac{1}{3} \end{aligned}$$

Conclusion: The z-transform does not exist, because the ROCs do not overlap.

$$\left\{z : |z| > \frac{1}{2}\right\} \cap \left\{z : |z| < \frac{1}{3}\right\} = \emptyset$$

Poles and Zeros

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

zeros are roots of numerator polynomial

$$\text{zeros} = \left\{0, \frac{1}{12}\right\}$$

poles are roots of denominator polynomial

$$\text{poles} = \left\{\frac{1}{2}, -\frac{1}{3}\right\}$$

To find poles and zeros, combine terms and express as rational function of z.

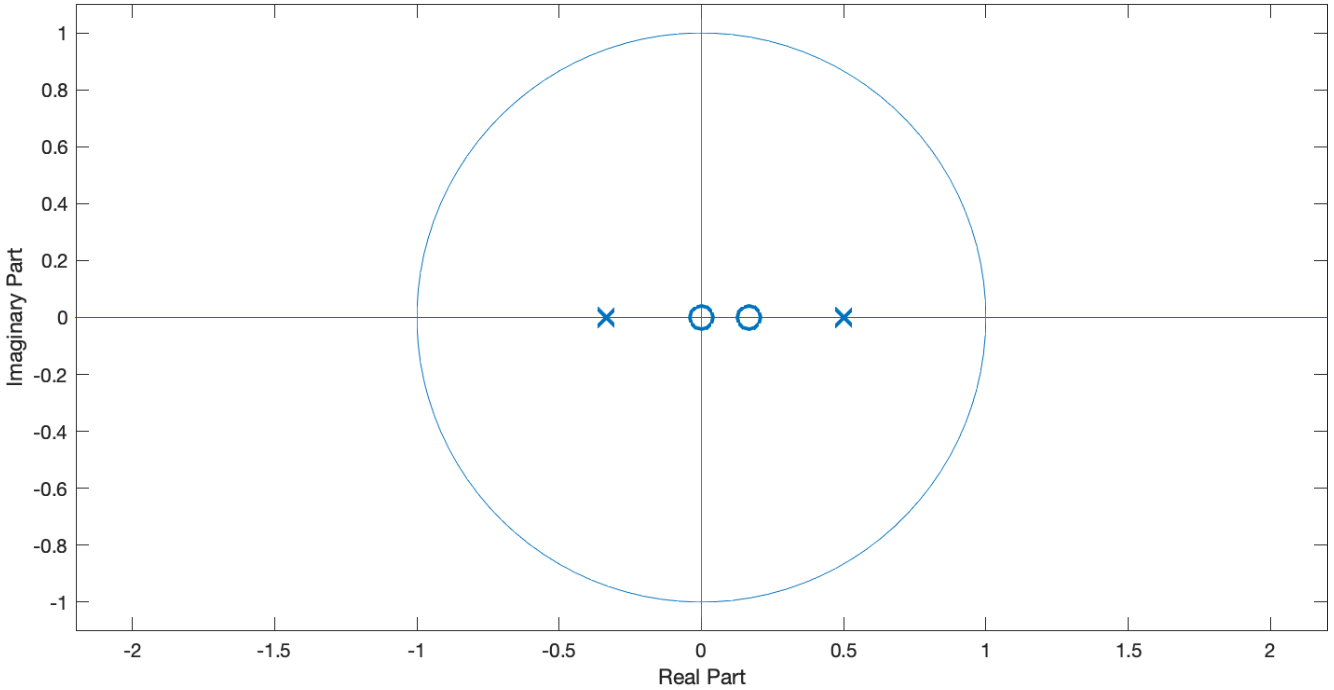
Poles and Zeros in Matlab

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

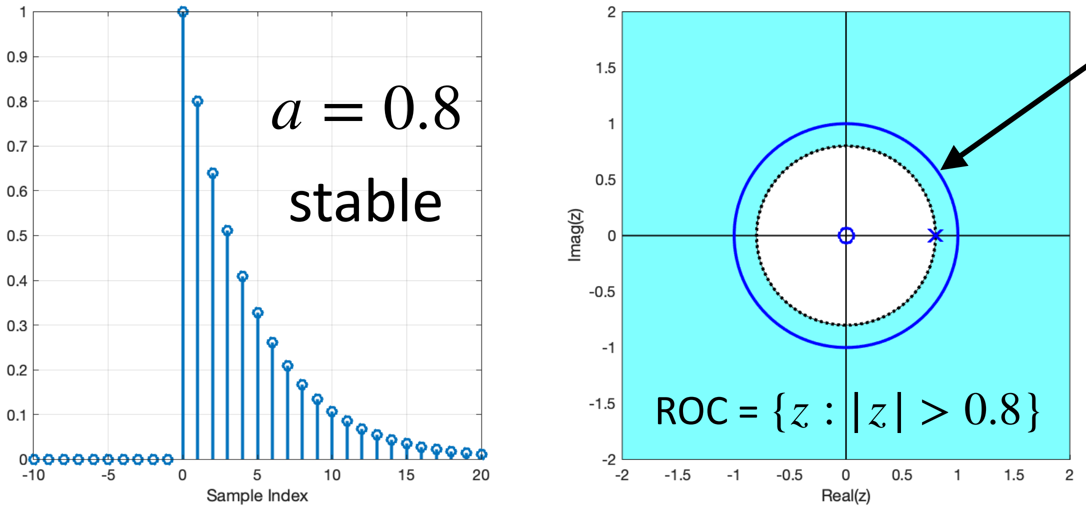
$$= \frac{2 - \frac{1}{6}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

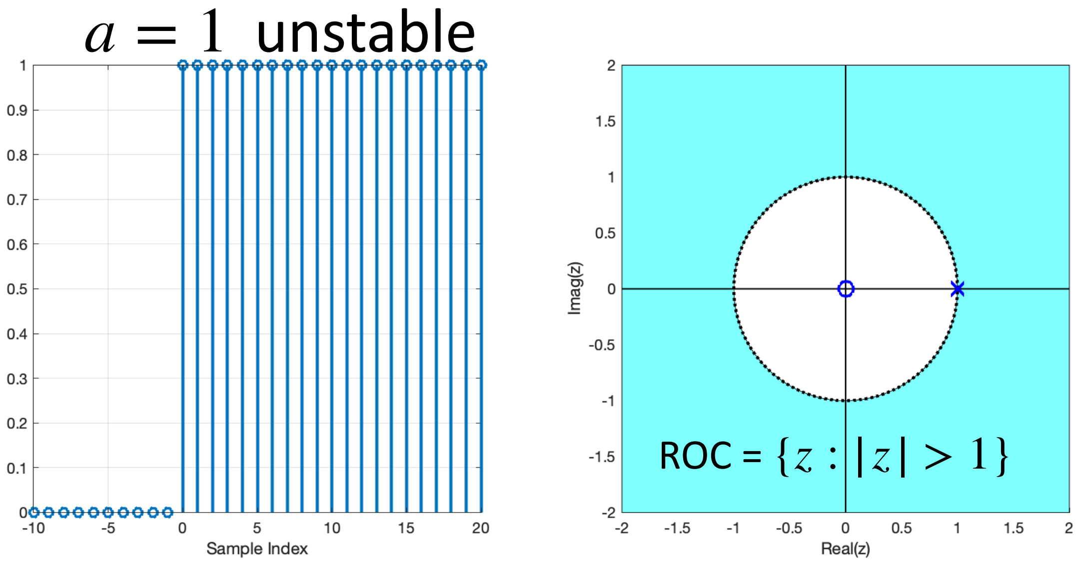
```
b = [2, -1/6];
a = [1, -1/6, -1/6];
zplane(b,a);
```

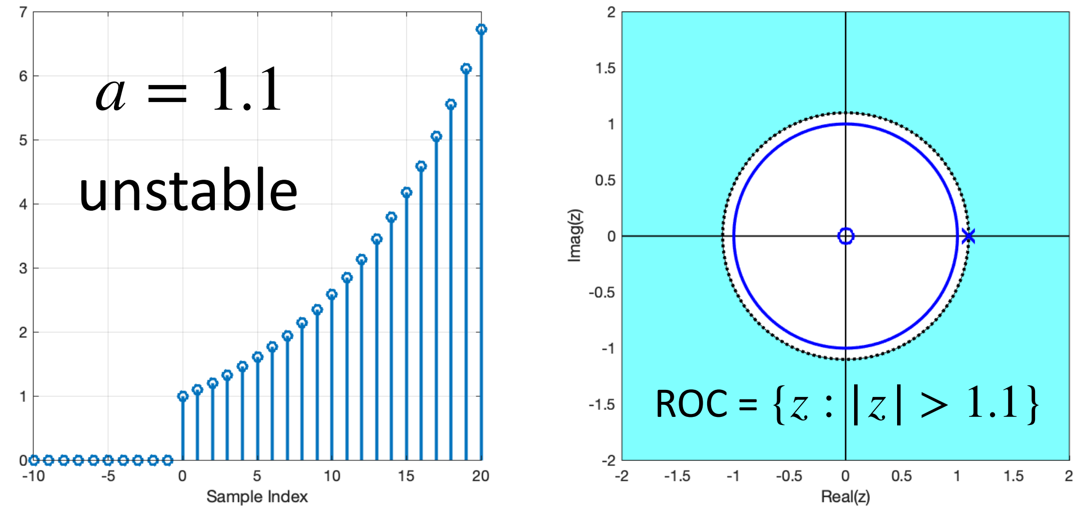


This is called a pole-zero plot.

Causal & Stable LTI Systems







unit circle (UC) = $\{z : |z| = 1\}$

$$X(f) = X(z)\big|_{z=e^{j2\pi f}}$$

For DTFT to exist, ROC must include UC.

X(z) is causal ⇒ ROC lies outside the outermost pole

X(z) is stable ⇒ ROC must include the unit circle

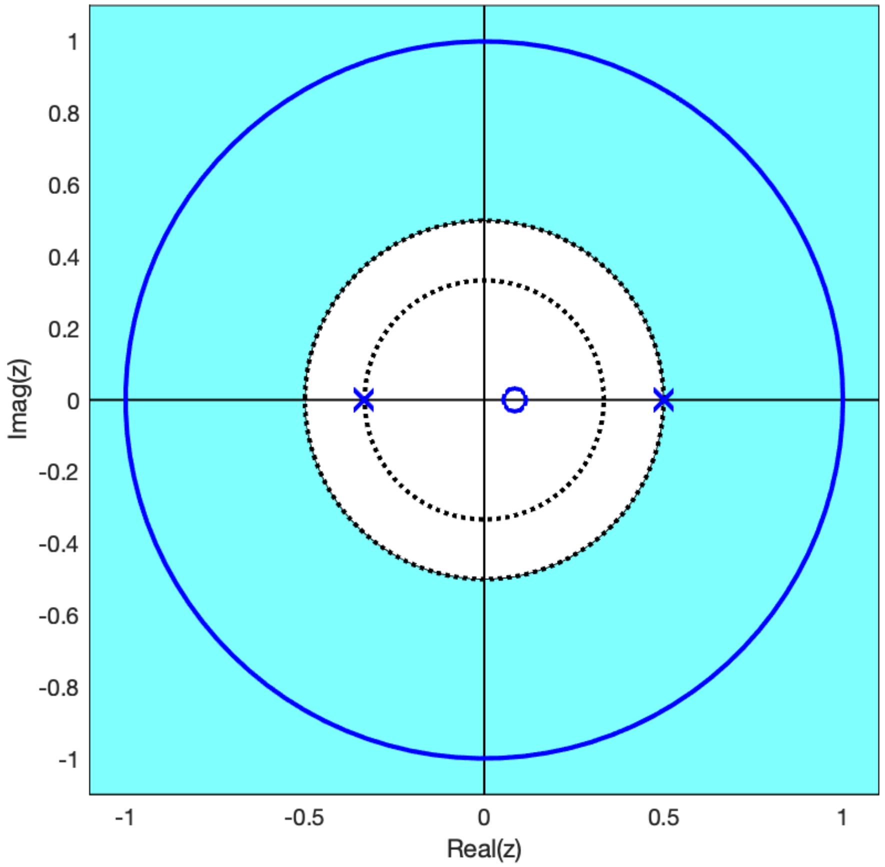
If X(z) is causal and stable, then all poles must lie inside the unit circle.

One $X(z)$ may correspond to many $x[n]$

$$X(z) = \frac{2 - \frac{1}{6}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

$$\text{ROC: } |z| > \frac{1}{2}$$



The ROC is outside each pole, so each pole contributes a causal term to x[n].

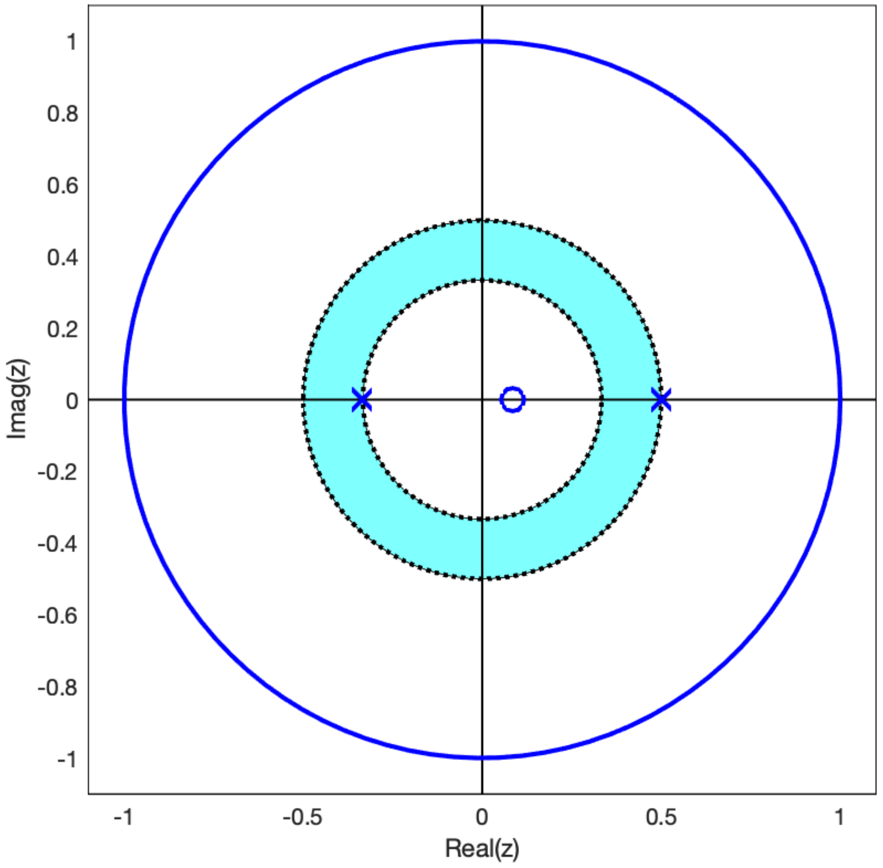
x[n] is causal (right-sided) signal.

One $X(z)$ may correspond to many $x[n]$

$$X(z) = \frac{2 - \frac{1}{6}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

ROC: $\frac{1}{3} < |z| < \frac{1}{2}$



The ROC is inside the pole at 1/2, so this pole contributes an anti-causal term to x[n].

The ROC is outside the pole at -1/3, so this pole contributes a causal term to x[n].

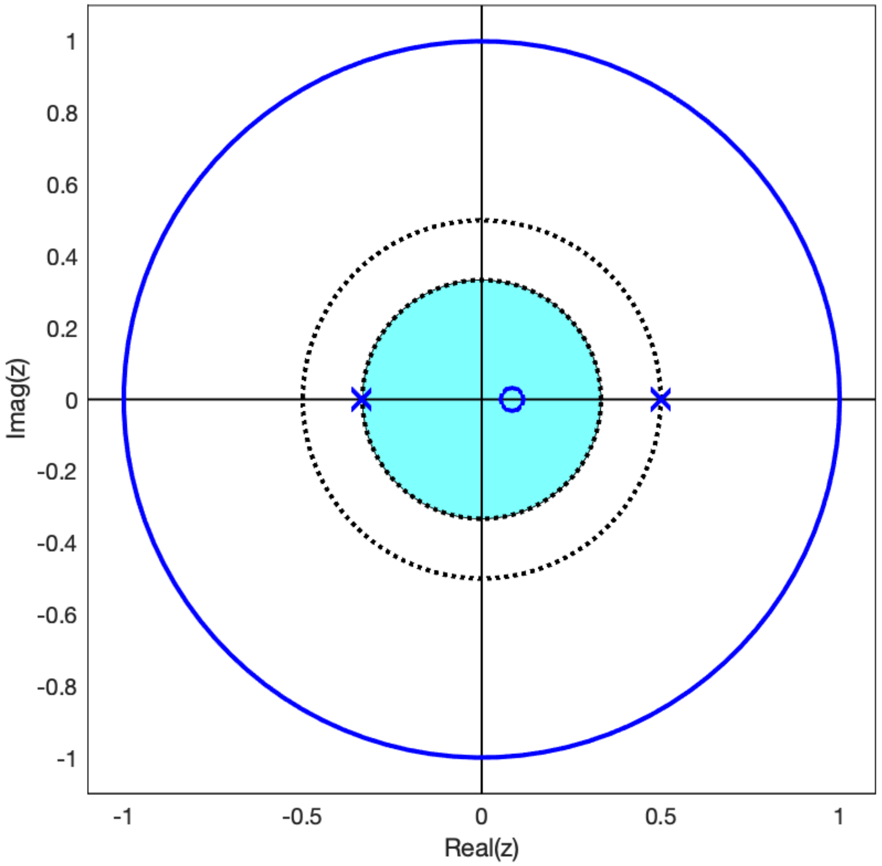
x[n] is a two-sided signal.

One $X(z)$ may correspond to many $x[n]$

$$X(z) = \frac{2 - \frac{1}{6}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

$\text{ROC: } |z| < \frac{1}{3}$



The ROC is inside each pole, so each pole contributes an anti-causal term to x[n].

x[n] is an anti-causal (left-sided) signal.

Another z-transform example

zeros at $z_k = a e^{j\frac{2\pi k}{M}}$
for $k = 0, 1, \cdots, M-1$

(causal finite-length signal)

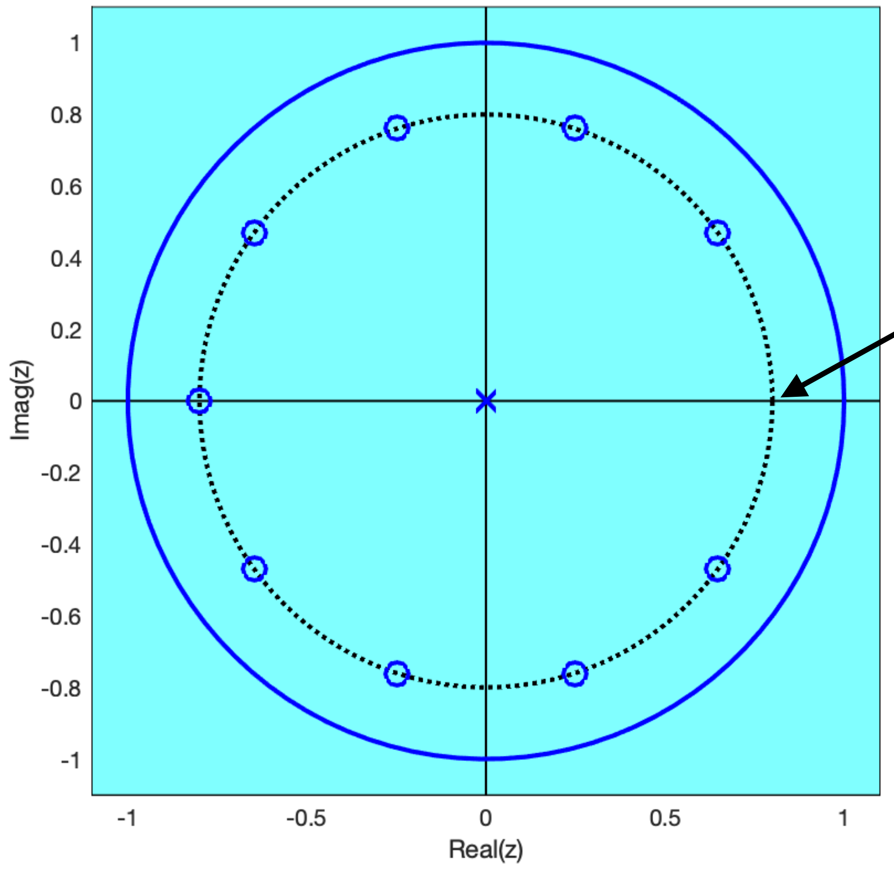
$$x[n] = a^n (u[n] - u[n-M])$$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} = \frac{1 - a^M z^{-M}}{1 - a z^{-1}} = \frac{1}{z^{(M-1)}} \frac{z^M - a^M}{z - a}$$

(M-1)th order
pole at z=0

pole at z=a

ROC: $|z| > 0$



pole-zero cancellation at z=a
(a=0.8 and N=10)

ROC is entire z-plane
except for the origin.

Another z-transform example

$$x[n] = a^n(u[n] - u[n-M])$$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} = \frac{1 - a^M z^{-M}}{1 - az^{-1}} = \frac{1}{z^{(M-1)}} \frac{z^M - a^M}{z - a}$$

Two ways to enter this into Matlab:

```
num = a.^[0:M-1];
den = 1;
```

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots + a^{M-1} z^{-(M-1)}$$

```
num = [1, zeros(1,M-1), -a^M];
den = [1, -a];
```

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

Table 3.1 (page 110) Common z-Transform Pairs

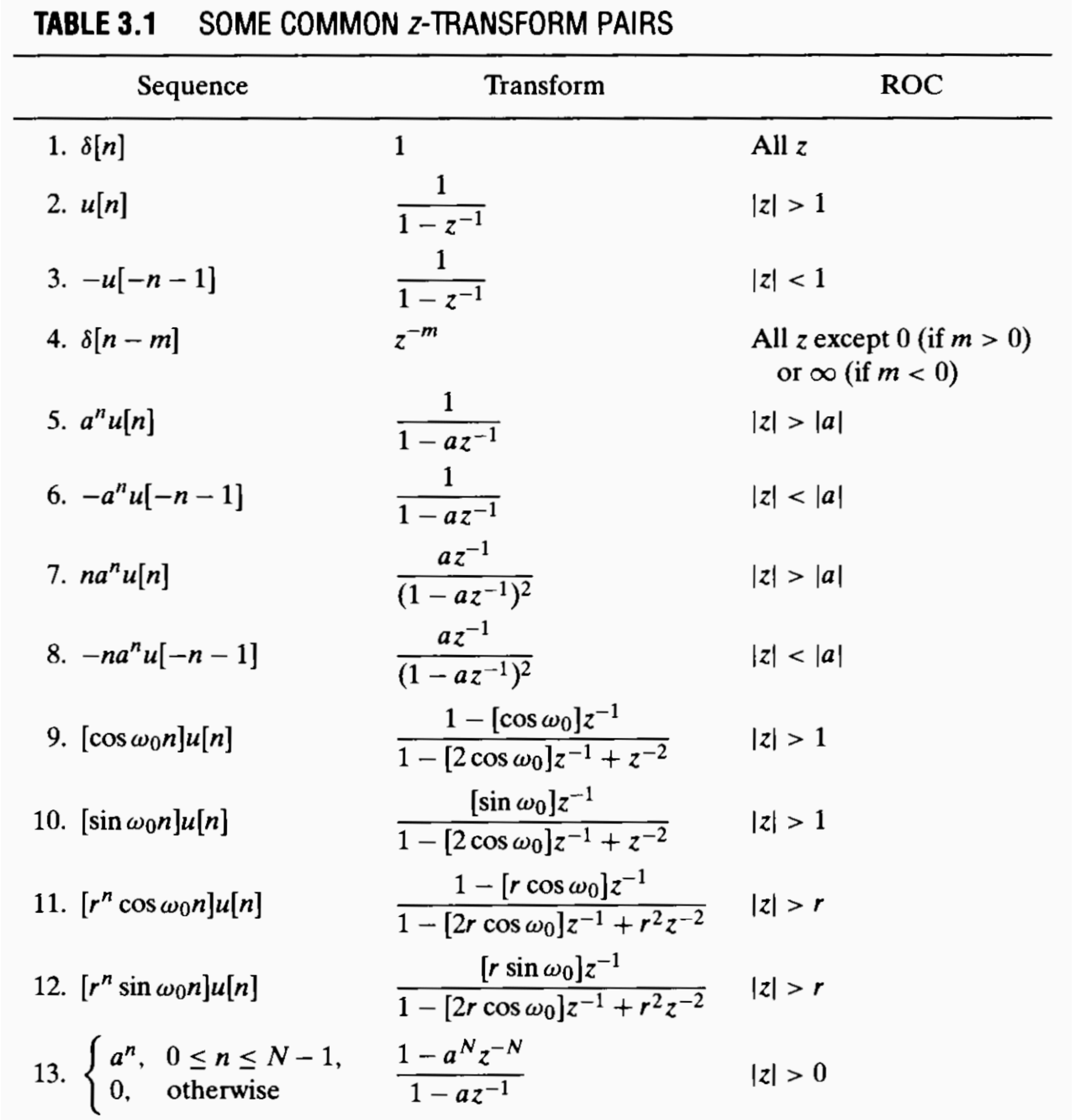


TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Note what is missing from the table: everlasting sin, cos, exp.

These signals do not have z-transforms.

Table 3.2 (page 132) Z-Transform Properties

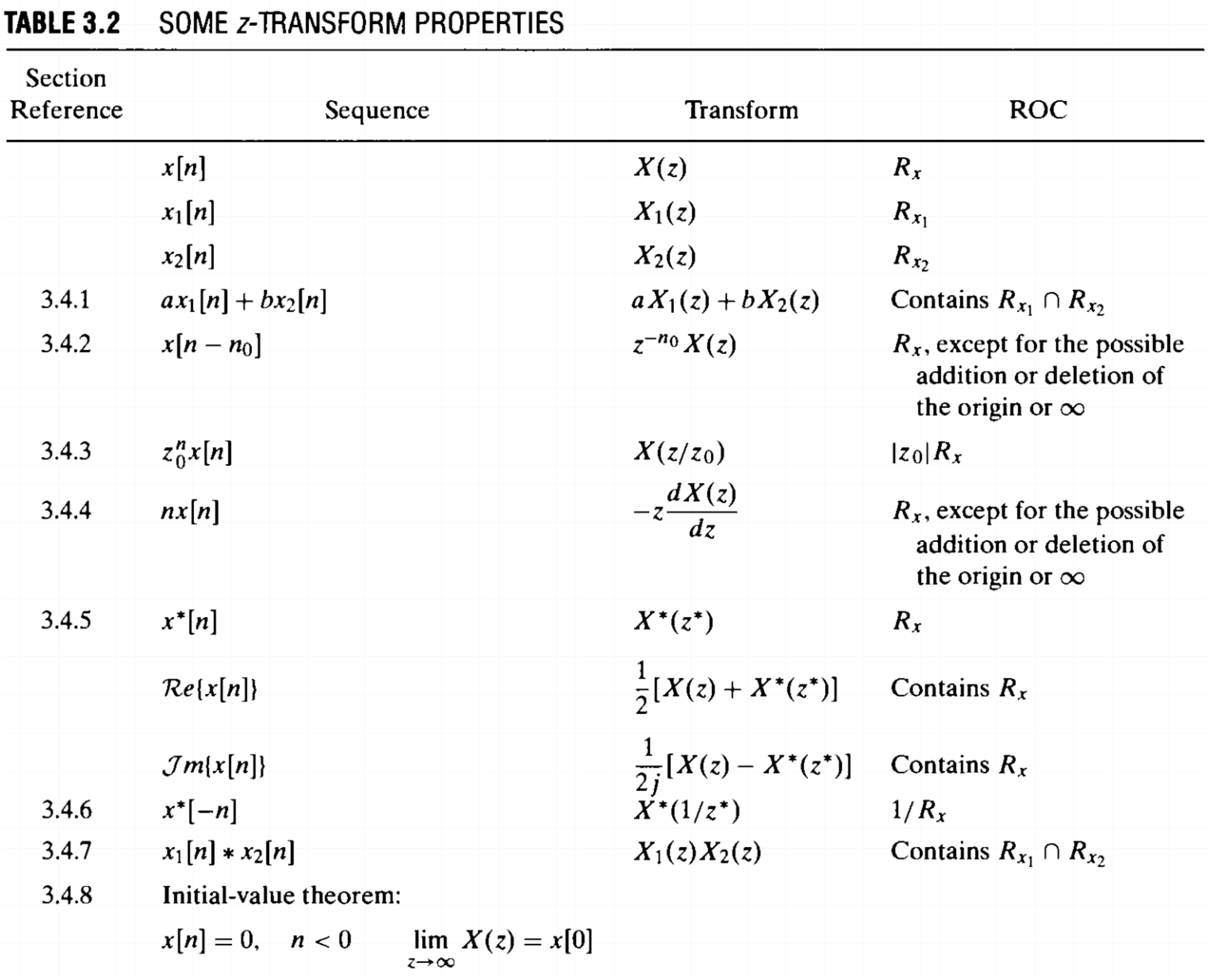


TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Section Reference	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x
	$x_1[n]$	$X_1(z)$	R_{x_1}
	$x_2[n]$	$X_2(z)$	R_{x_2}
3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
	$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.8	Initial-value theorem:		
	$x[n] = 0, \quad n < 0$	$\lim_{z \to \infty} X(z) = x[0]$	

These are easy to derive, but take care of the ROCs.

Initial & Final Value Theorems

Initial Value Theorem: If $x[n]$ is causal, then

$$\lim_{z\to\infty} X(z) = x[0].$$

Final Value Theorem: If $x[n]$ is causal, then

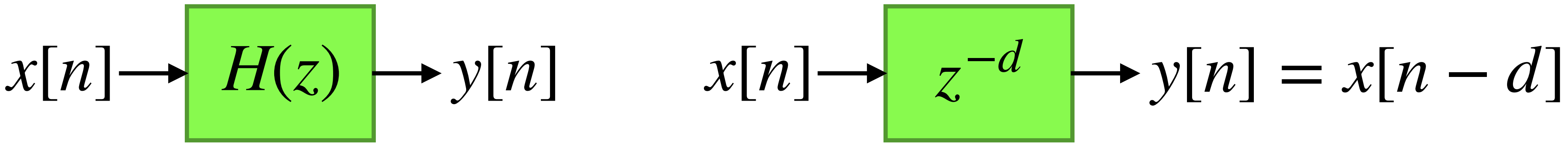
$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(1 - z^{-1})X(z).$$

Impulse Response and Transfer Functions

The z-transform of the impulse response $h[n]$ is the transfer function $H(z)$.

$$h[n] = \delta[n-d] \quad \longleftrightarrow \quad H(z) = z^{-d}$$

$$\text{ROC}: \begin{cases} \text{whole } z\text{-plane}, & d = 0, \quad \text{(identity system)} \\ |z| > 0, & d > 0, \quad \text{(delay)} \\ |z| < 0, & d < 0, \quad \text{(advance)} \end{cases}$$



These are common examples of signal processing block diagrams.

Transfer Functions of Difference Equations (LTI Systems)

Time-shift property: $x[n-d] \quad \Longleftrightarrow \quad X(z)z^{-d}$

Difference eqn.: $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Take z-transform: $$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Transfer Functions of Difference Equations (LTI Systems)

Time-shift property: $x[n-d] \quad \Longleftrightarrow \quad X(z)z^{-d}$

Difference eqn.: $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Take z-transform: $$\sum_{k=0}^{N} a_k z^{-k}Y(z) = \sum_{k=0}^{M} b_k z^{-k}X(z)$$

Solve for $H(z)$: $$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$$B(z) = \sum_{k=0}^{M} b_k z^{-k} \qquad\qquad A(z) = \sum_{k=0}^{N} a_k z^{-k}$$

Transfer Functions of Difference Equations (LTI Systems)

$$\text{Time-shift property:} \quad x[n-d] \quad \Longleftrightarrow \quad X(z)z^{-d}$$

$$\text{Difference eqn.:} \quad \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

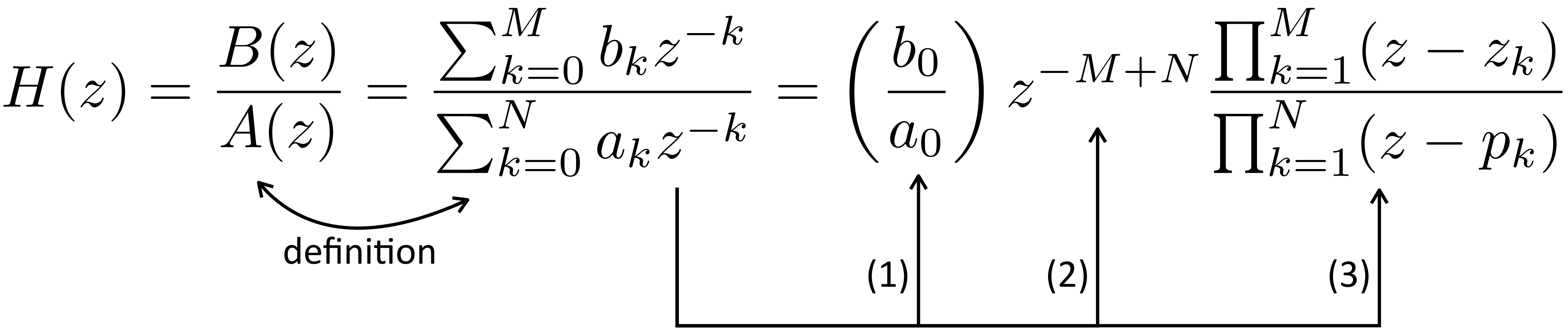
$$\text{Take } z\text{-transform:} \quad \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\text{Solve for } H(z)\text{:} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

Given a difference equation, the transfer function H(z) may be written down by inspection.

Given a rational transfer function H(z)=B(z)/A(z), the difference equation may be written down by inspection.

Transfer Functions of Difference Equations (LTI Systems)



$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) z^{-M+N} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

(1) Factor out b_0 from numerator and a_0 from denominator

(2) Factor out z^{-M} from numerator and z^{-N} from denominator

(3) Factor resulting polynomials

Roots of numerator polynomial are zeros $z_k, k = 0, 1, \cdots, M$

Roots of denominator polynomial are poles $p_k, k = 0, 1, \cdots, N$

$\max\{0, N - M\}$ zeros at $z = 0$ (if $N > M$)

$\max\{0, M - N\}$ poles at $z = 0$ (if $M > N$)

Poles or zeros may also occur at $z = \infty$

Poles at $z = \infty$ if $X(\infty) = \infty$

Zeros at $z = \infty$ if $X(0) = \infty$

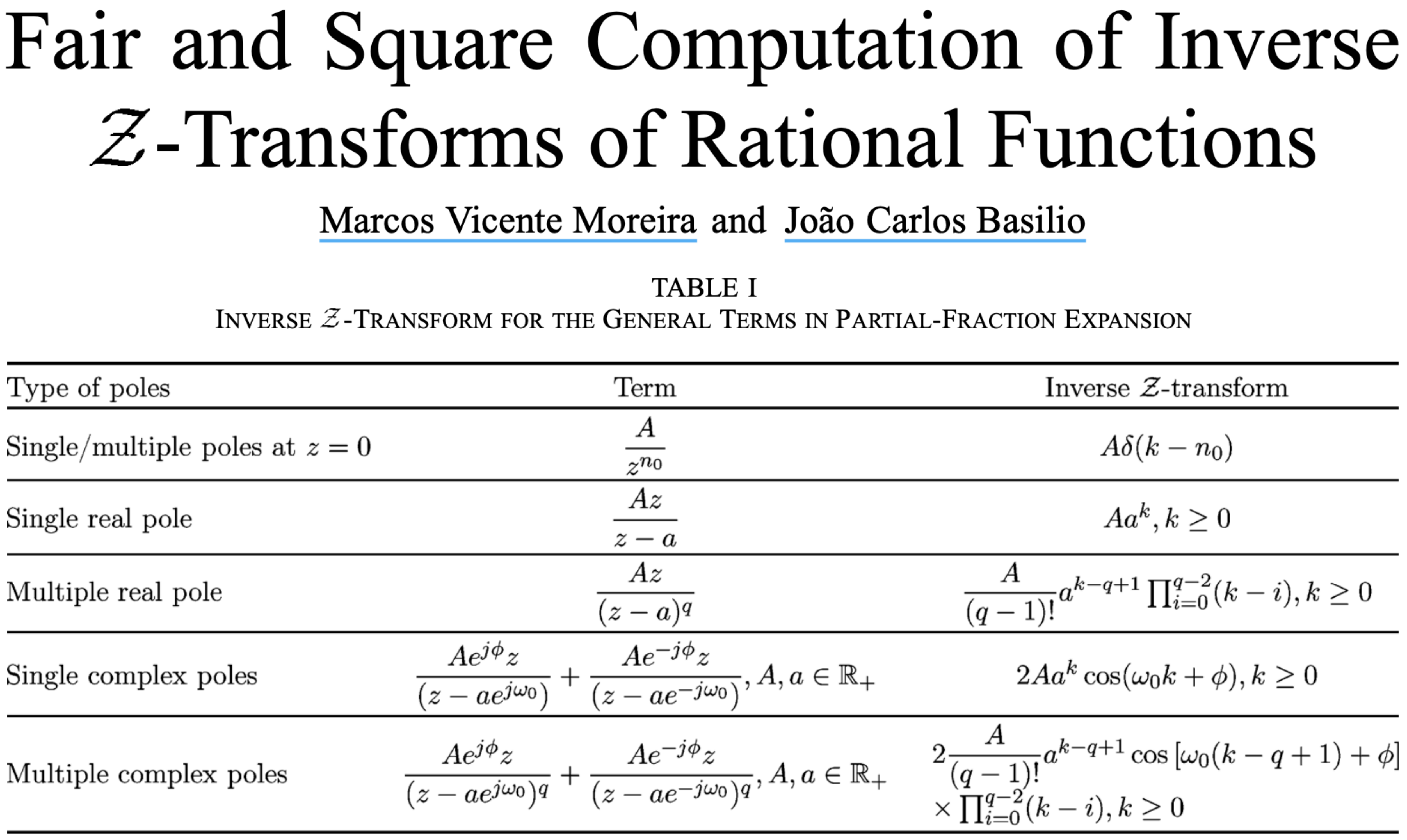
Equal numbers of poles and zeros (counting $z = \infty$)

Transfer Functions of Difference Equations (LTI Systems)

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} = \left(\frac{b_0}{a_0}\right) z^{-M+N} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

There are M finite zeros and N finite poles.

Inverse Z-Transform



Fair and Square Computation of Inverse $\mathcal{Z}$-Transforms of Rational Functions

Marcos Vicente Moreira and João Carlos Basilio

TABLE I

INVERSE $\mathcal{Z}$-TRANSFORM FOR THE GENERAL TERMS IN PARTIAL-FRACTION EXPANSION

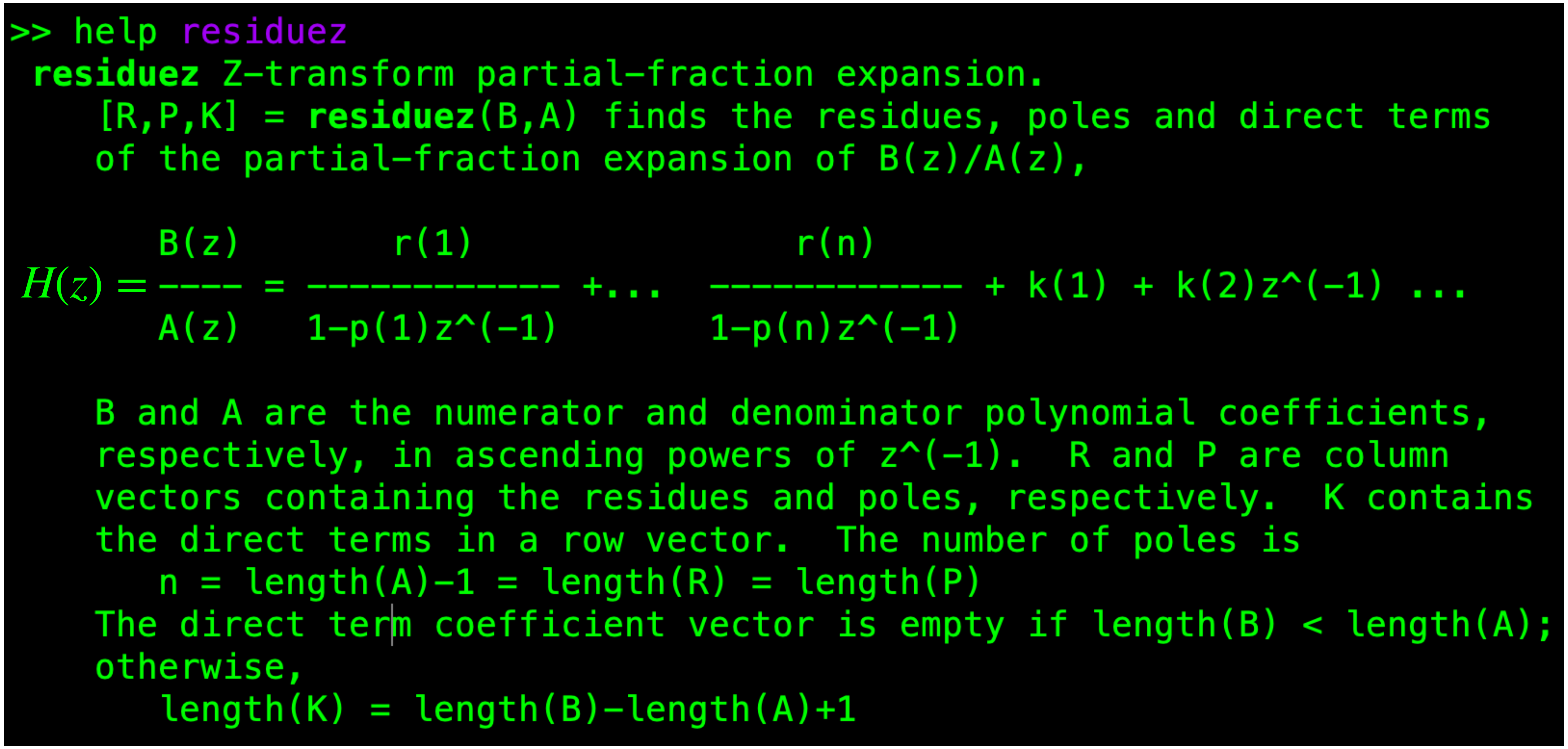
Type of poles	Term	Inverse $\mathcal{Z}$-transform
Single/multiple poles at $z = 0$	$\dfrac{A}{z^{n_0}}$	$A\delta(k - n_0)$
Single real pole	$\dfrac{Az}{z-a}$	$Aa^k, k \geq 0$
Multiple real pole	$\dfrac{Az}{(z-a)^q}$	$\dfrac{A}{(q-1)!} a^{k-q+1} \prod_{i=0}^{q-2}(k-i), k \geq 0$
Single complex poles	$\dfrac{Ae^{j\phi}z}{(z-ae^{j\omega_0})} + \dfrac{Ae^{-j\phi}z}{(z-ae^{-j\omega_0})}, A, a \in \mathbb{R}_+$	$2Aa^k \cos(\omega_0 k + \phi), k \geq 0$
Multiple complex poles	$\dfrac{Ae^{j\phi}z}{(z-ae^{j\omega_0})^q} + \dfrac{Ae^{-j\phi}z}{(z-ae^{-j\omega_0})^q}, A, a \in \mathbb{R}_+$	$2\dfrac{A}{(q-1)!} a^{k-q+1} \cos\left[\omega_0(k-q+1)+\phi\right]$ $\times \prod_{i=0}^{q-2}(k-i), k \geq 0$

Paper is linked on "resources" tab of class web site.

Expand $H(z)$ in partial fraction expansion, then write down $h[n]$.

Inverse Z-Transform via Computer

You have already encountered PFE in calculus and in continuous-time signals and systems. This semester you are going to learn how to do PFE on computer (Matlab).



```
>> help residuez
 residuez Z-transform partial-fraction expansion.
    [R,P,K] = residuez(B,A) finds the residues, poles and direct terms
    of the partial-fraction expansion of B(z)/A(z),

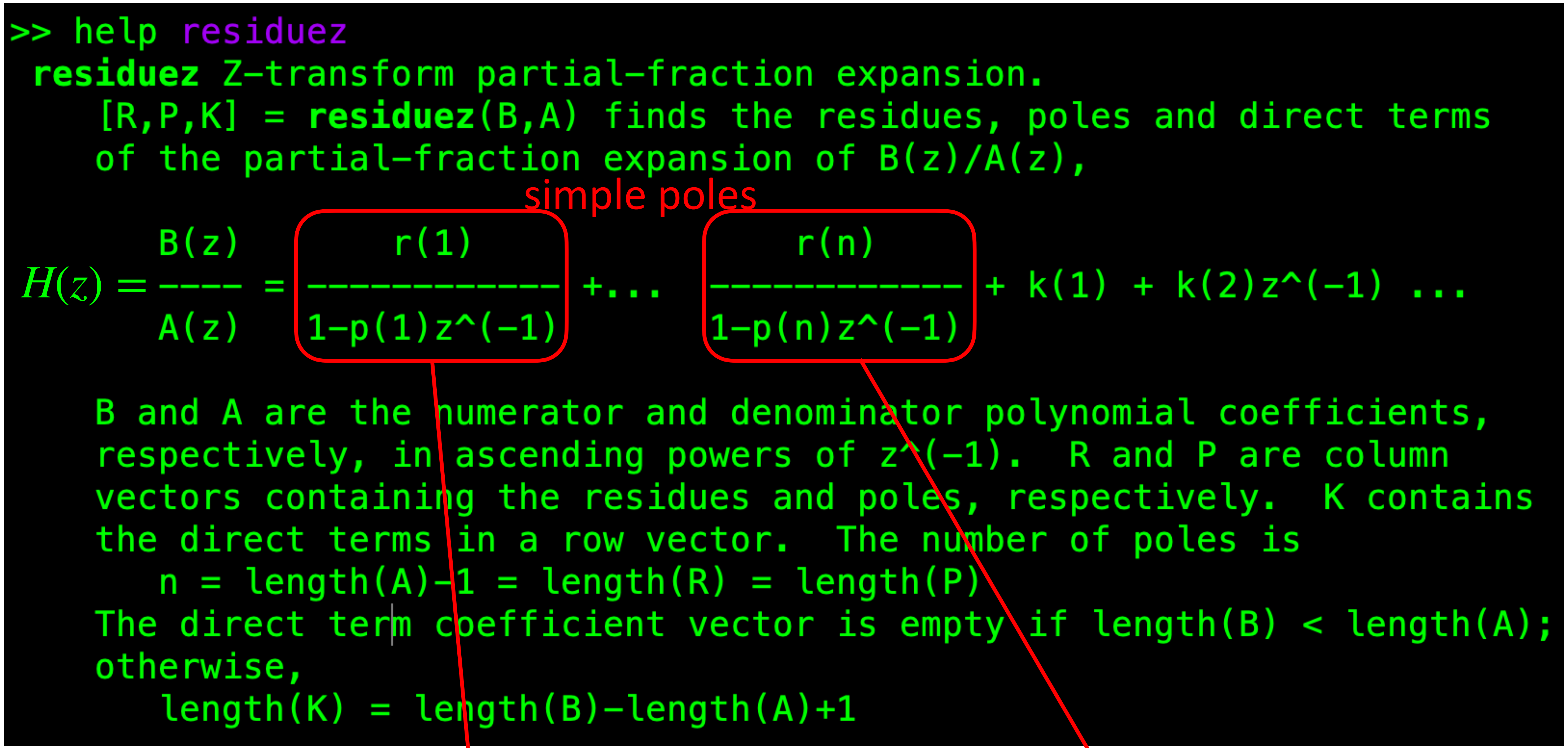
       B(z)       r(1)                 r(n)
H(z) = ---- = ------------ +...   ------------ + k(1) + k(2)z^(-1) ...
       A(z)   1-p(1)z^(-1)        1-p(n)z^(-1)

    B and A are the numerator and denominator polynomial coefficients,
    respectively, in ascending powers of z^(-1).  R and P are column
    vectors containing the residues and poles, respectively.  K contains
    the direct terms in a row vector.  The number of poles is
       n = length(A)-1 = length(R) = length(P)
    The direct term coefficient vector is empty if length(B) < length(A);
    otherwise,
       length(K) = length(B)-length(A)+1
```

$$
\begin{aligned}
h[n] = {} & r(1)[p(1)]^n u[n] + \cdots + r(n)[p(n)]^n u[n] \\
& + k(1)\delta[n] + k(2)\delta[n-1] + \cdots
\end{aligned}
$$

Inverse Z-Transform via Computer

Note: In what follows, we are always assuming that the ROC is the region outside the outermost pole. The inverse z-transform, i.e. the impulse response h[n], will be a causal sequence.



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       A(z)    1-p(1)z^(-1)         1-p(n)z^(-1)

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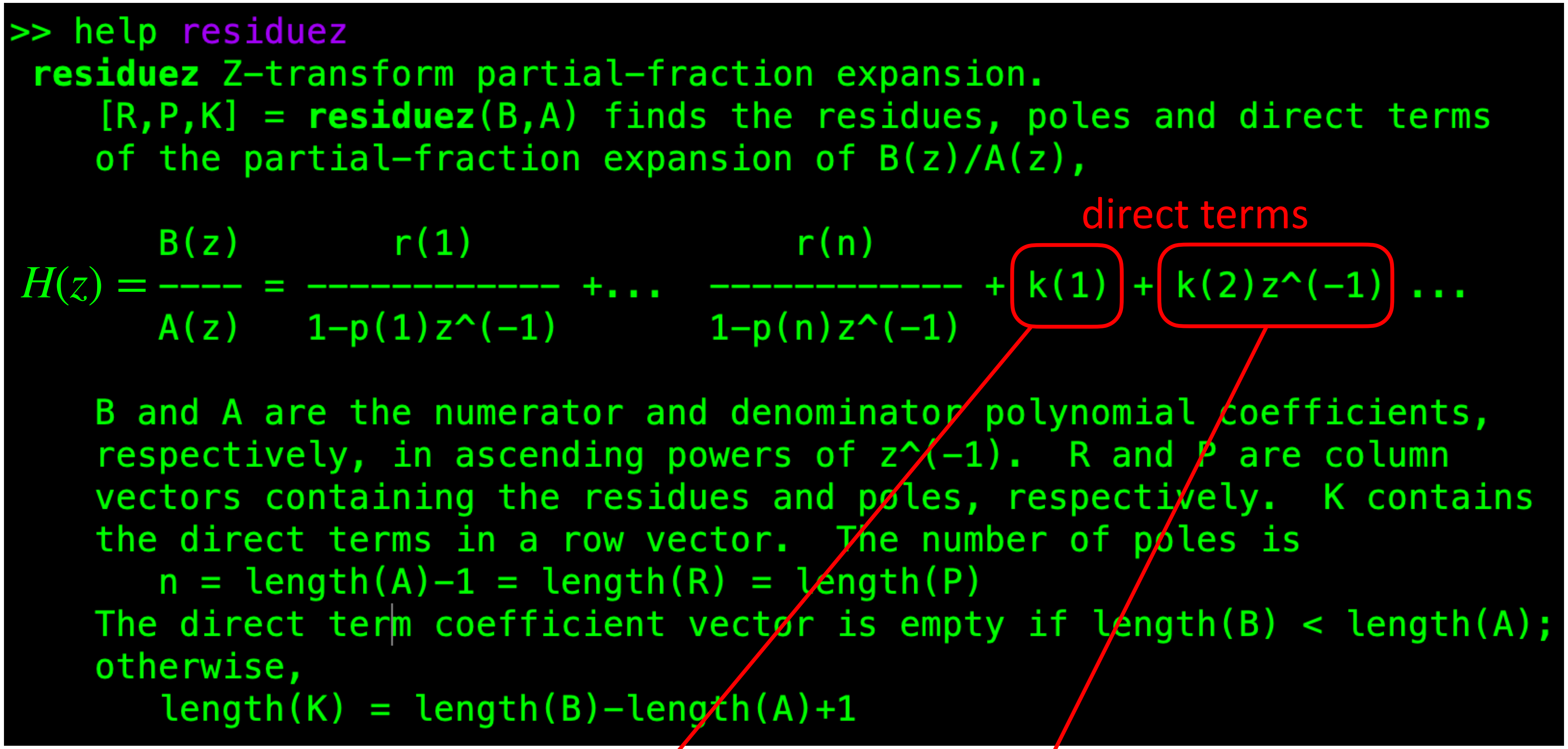
    otherwise,
       length(K) = length(B)-length(A)+1
```

$$h[n] = \boxed{r(1)[p(1)]^n u[n]} + \cdots + \boxed{r(n)[p(n)]^n u[n]}$$
$$+ k(1)\delta[n] + k(2)\delta[n-1] + \cdots$$

Inverse Z-Transform via Computer

Note: In what follows, we are always assuming that the ROC is the region outside the outermost pole. The inverse z-transform, i.e. the impulse response h[n], will be a causal sequence.



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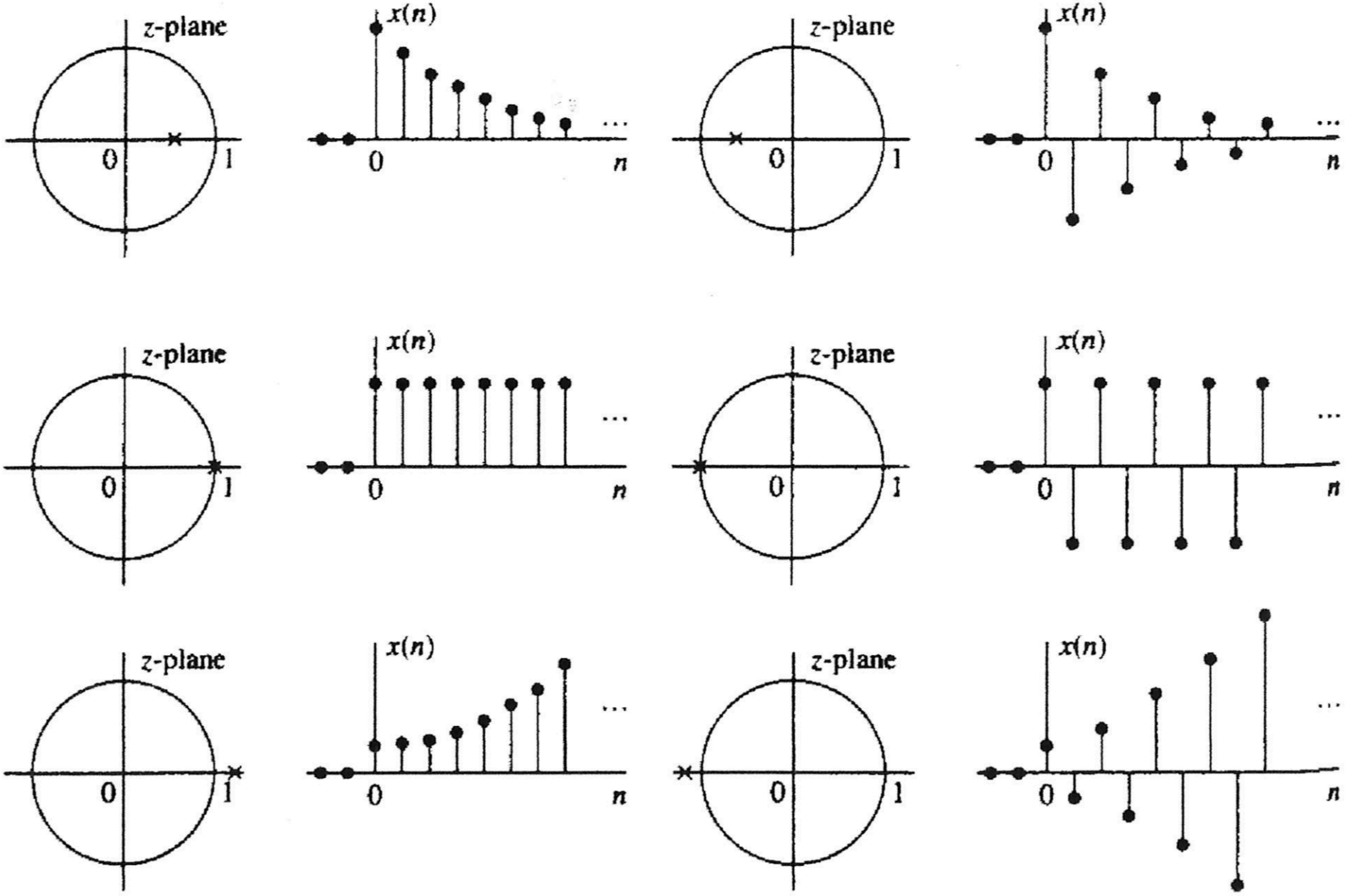
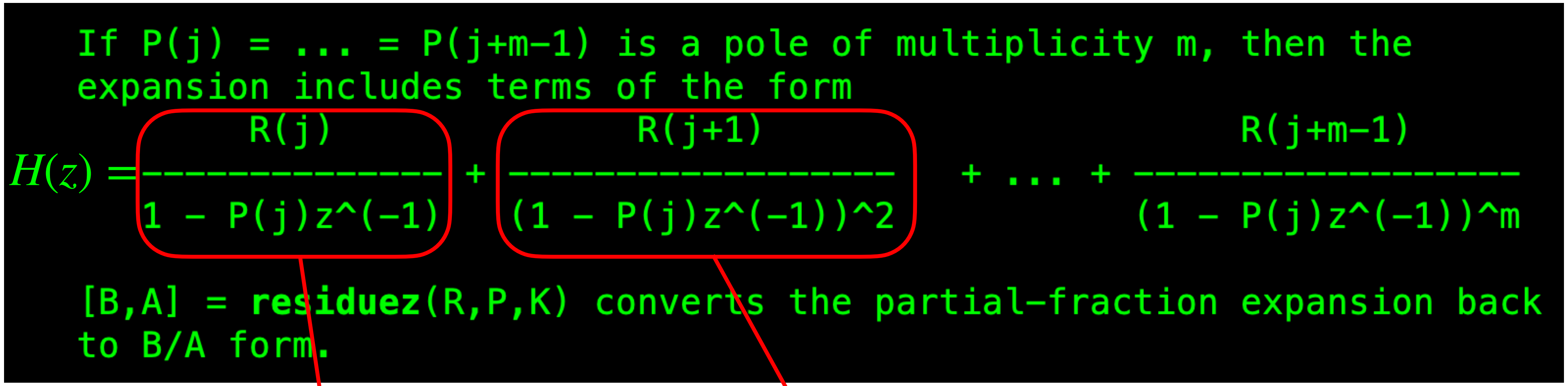
       length(K) = length(B)-length(A)+1
```

$$h[n] = r(1)[p(1)]^n u[n] + \cdots + r(n)[p(n)]^n u[n]$$
$$+ \boxed{k(1)\delta[n]} + \boxed{k(2)\delta[n-1]} + \cdots$$

Inverse Z-Transform Components: 1st Order Poles



Convergence and Regions of Convergence



```
    If P(j) = ... = P(j+m-1) is a pole of multiplicity m, then the
    expansion includes terms of the form
                R(j)                   R(j+1)                          R(j+m-1)
H(z) = --------------- + -------------------- + ... + ------------------
       1 - P(j)z^(-1)    (1 - P(j)z^(-1))^2           (1 - P(j)z^(-1))^m

    [B,A] = residuez(R,P,K) converts the partial-fraction expansion back
    to B/A form.
```

$$h[n] = \boxed{R(j)[P(j)]^n u[n]} + \boxed{R(j+1)(n+1)[P(j)]^n u[n]}$$

In this class, we will only encounter second order poles,
i.e. poles with multiplicity 2.

For higher order poles, see the paper …

Fair and Square Computation of Inverse
$\mathcal{Z}$-Transforms of Rational Functions

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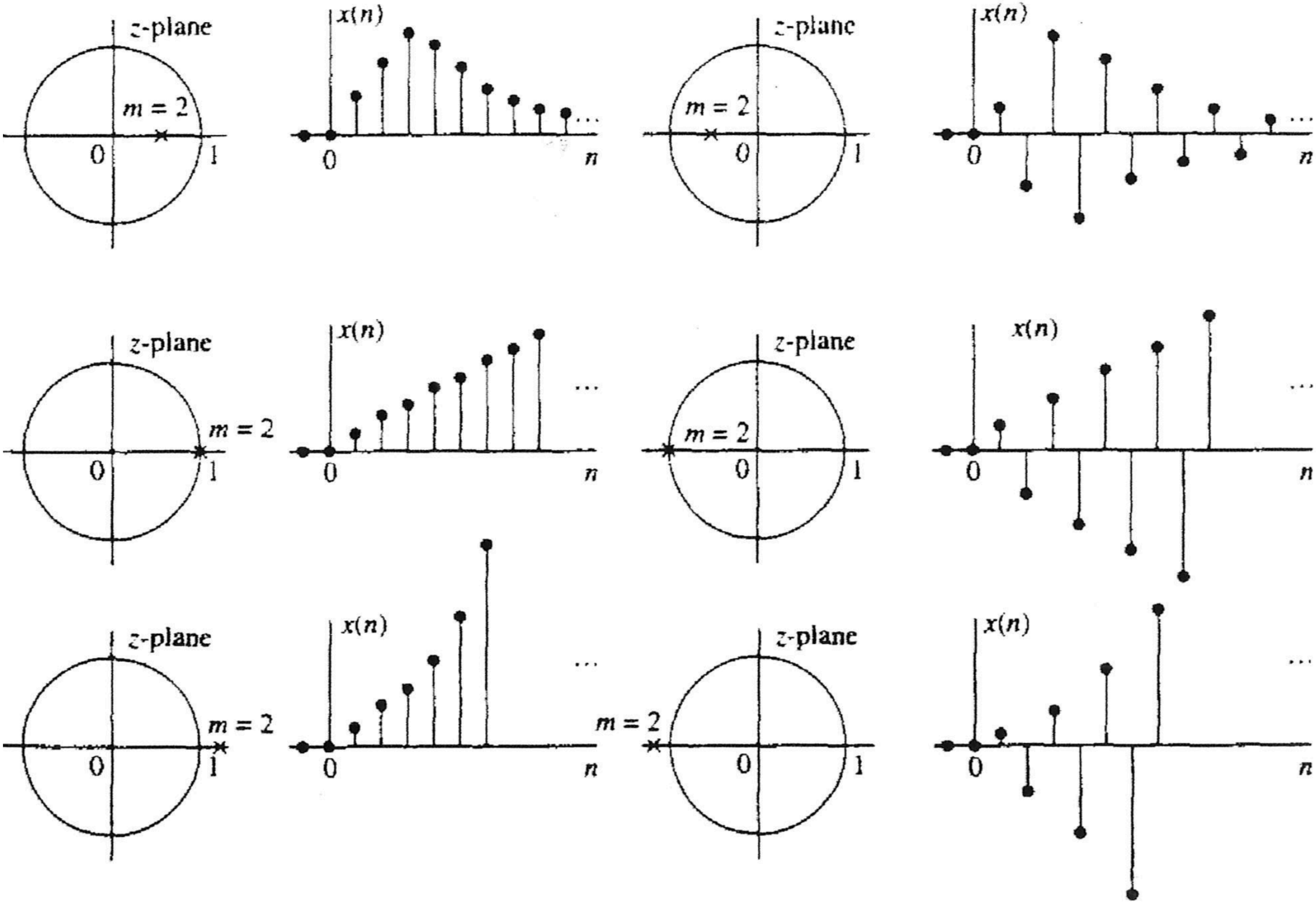
Inverse Z-Transform Components: 2nd Order Poles



Inverse Z-Transform: Complex Conjugate Pair of Poles

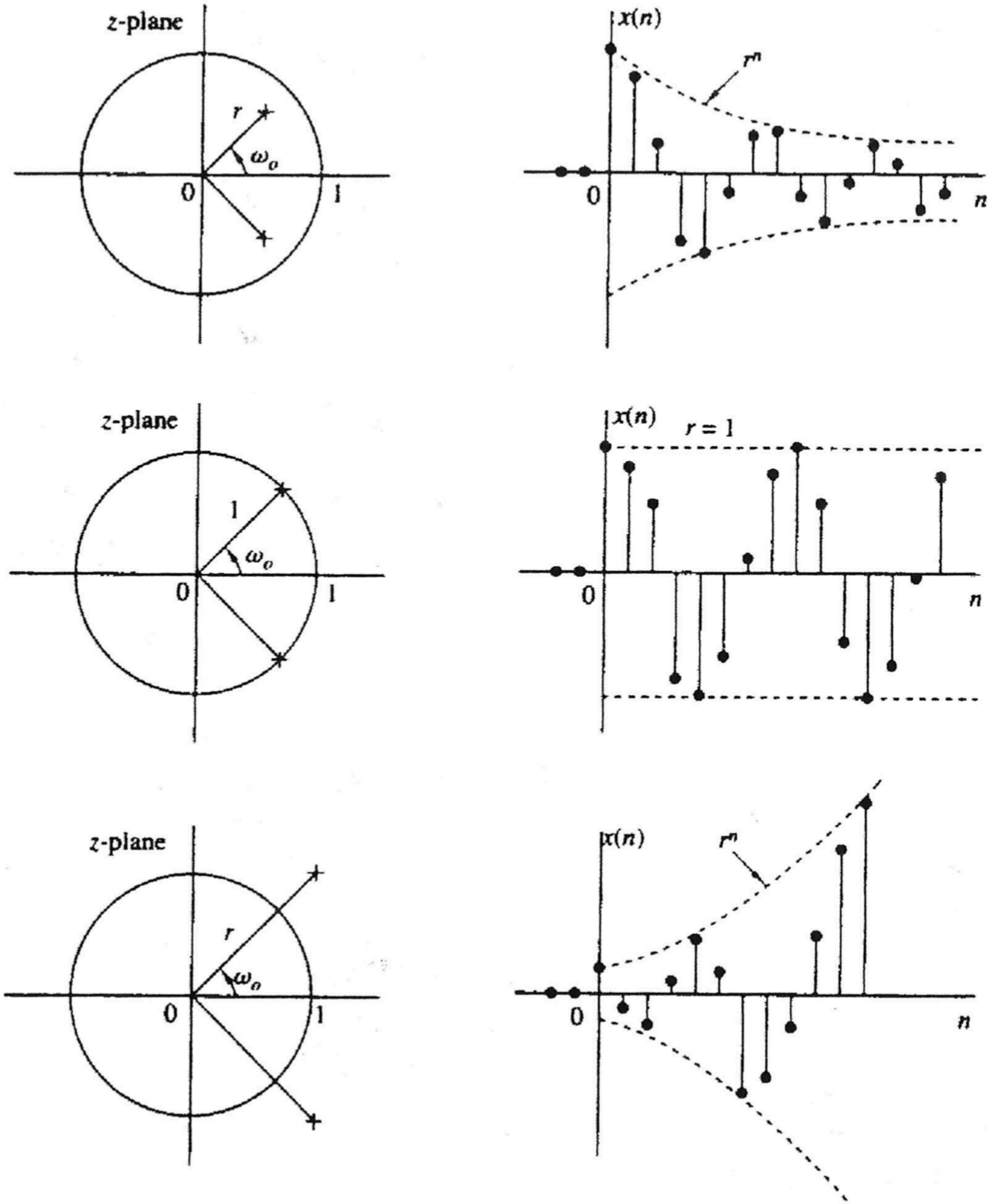
One other common case is a pair of complex conjugate poles.

$$H(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

$$\begin{aligned} h[n] &= \left(Ap^n + A^* (p^*)^n\right) u[n] \\ &= 2|A| r^n \cos(2\pi f_0 n + \theta) u[n] \end{aligned}$$

$$A = |A| e^{j\theta}, \quad p = re^{j2\pi f_0}$$

Inverse Z-Transform Components: Complex Conjugate Pole Pairs



Some Useful Matlab Functions

polynomial multiplication: conv([1,-1],[1,-1/2]) = [1, -1.5, 0.5]

$$(1-z^{-1})(1-0.5z^{-1}) = 1 - 1.5z^{-1} + 0.5z^{-2}$$

polynomial synthesis from roots: poly([1,0.5]) = [1, -1.5, 0.5]

$$(1-z^{-1})(1-0.5z^{-1}) = 1 - 1.5z^{-1} + 0.5z^{-2}$$

root a polynomial (factor polynomial): roots([1, -1.5, 0.5]) = [1, 0.5]

$$1 - 1.5z^{-1} + 0.5z^{-2} = (1-z^{-1})(1-0.5z^{-1})$$

$H(z)$

partial fraction expansion: b=[1,1]; a=poly([1,0.5]); [r,p,k]=residuez(b,a);

r = [4, -3]; p = [1, 0.5]; k=[];

$$H(z) = \frac{1+z^{-1}}{(1-z^{-1})(1-0.5z^{-1})} = \frac{1+z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= \frac{4}{1-(1)z^{-1}} + \frac{-3}{1-0.5z^{-1}}$$

$$h[n] = 4(1)^n u[n] + (-3)(0.5)^n u[n]$$

Convergence and Regions of Convergence

Transfer function H(z) = B(z)/A(z) is represented in Matlab by vectors **b** and **a** containing the coefficients of B(z) and A(z), respectively.

make a pole-zero plot: `zplane(b,a)`

plot the frequency response: `freqz(b,a)`

filter a signal: `y = filter(b,a,x)`

compute the first 100 samples of the impulse response:

```
h = filter(b,a,[1,zeros(1,99)]);
```

compute analytical expression for the impulse response:

```
[r,p,k] = residuez(b,a);
```

Now write down h[n] from the residues, poles, and direct terms.

Inverse Z-Transform Example

$$y[n] = 0.5y[n-1] + x[n] \quad \Rightarrow \quad y[n] - 0.5y[n-1] = x[n]$$

$$H(z) = \frac{1}{1 - 0.5z^{-1}}, |z| > 0.5 \quad \Rightarrow \quad h[n] = (0.5)^n u[n]$$

$$x[n] = 10\cos\left(\frac{\pi n}{4}\right) u[n] \quad \Rightarrow \quad X(z) = \frac{10\left(1 - \frac{1}{\sqrt{2}} z^{-1}\right)}{1 - \sqrt{2} z^{-1} + z^{-2}}, |z| > 1$$

$$y[n] = x[n] * h[n] \quad \Rightarrow \quad Y(z) = X(z)H(z)$$

$$Y(z) = \frac{10\left(1 - \frac{1}{\sqrt{2}} z^{-1}\right)}{(1 - 0.5z^{-1})(1 - e^{j\pi/4} z^{-1})(1 - e^{-j\pi/4} z^{-1})}$$

$$\mathrm{ROC}_Y = \mathrm{ROC}_X \cap \mathrm{ROC}_H = \{z : |z| > 1\}$$

```
[r,p,k]=residuez(10*[1,-1/sqrt(2)],conv([1,-0.5],[1,-sqrt(2),1]));
```

Inverse Z-Transform Example

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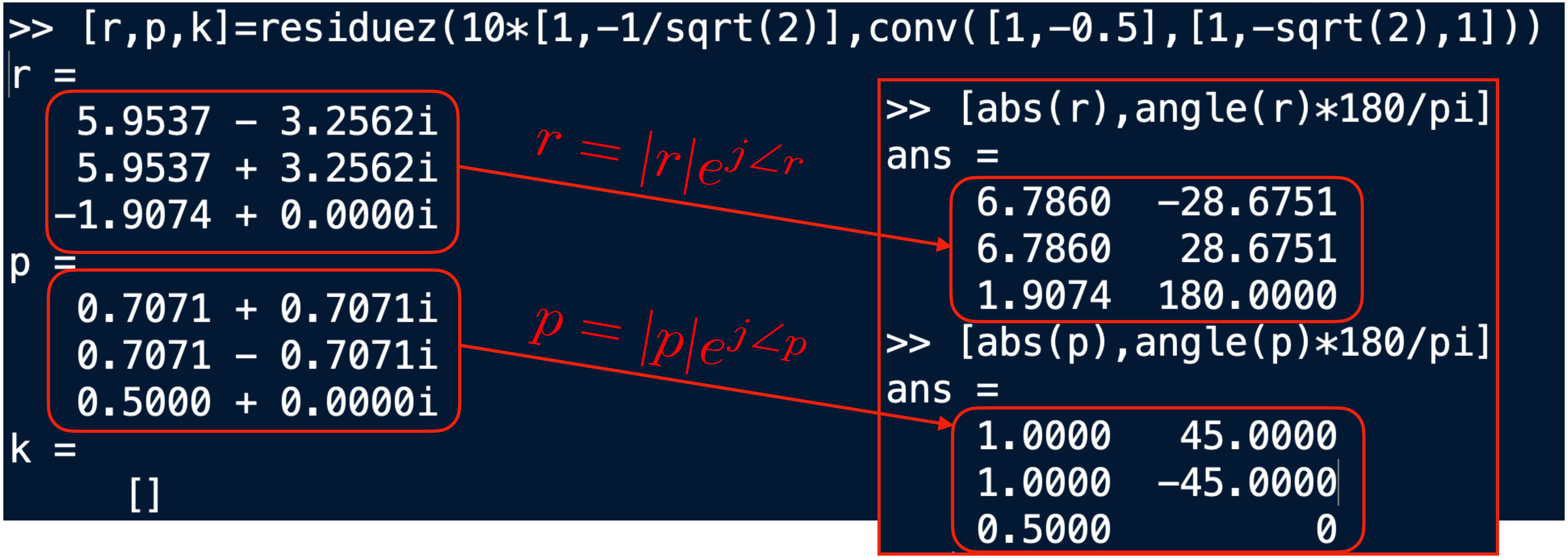
```
[r,p,k]=residuez(10*[1,-1/sqrt(2)],conv([1,-0.5],[1,-sqrt(2),1]));
```

Inverse Z-Transform Example (continued)



```
>> [r,p,k]=residuez(10*[1,-1/sqrt(2)],conv([1,-0.5],[1,-sqrt(2),1]))
r =
   5.9537 - 3.2562i
   5.9537 + 3.2562i
  -1.9074 + 0.0000i
p =
   0.7071 + 0.7071i
   0.7071 - 0.7071i
   0.5000 + 0.0000i
k =
     []
```

```
>> [abs(r),angle(r)*180/pi]
ans =
    6.7860  -28.6751
    6.7860   28.6751
    1.9074  180.0000
>> [abs(p),angle(p)*180/pi]
ans =
    1.0000   45.0000
    1.0000  -45.0000
    0.5000         0
```

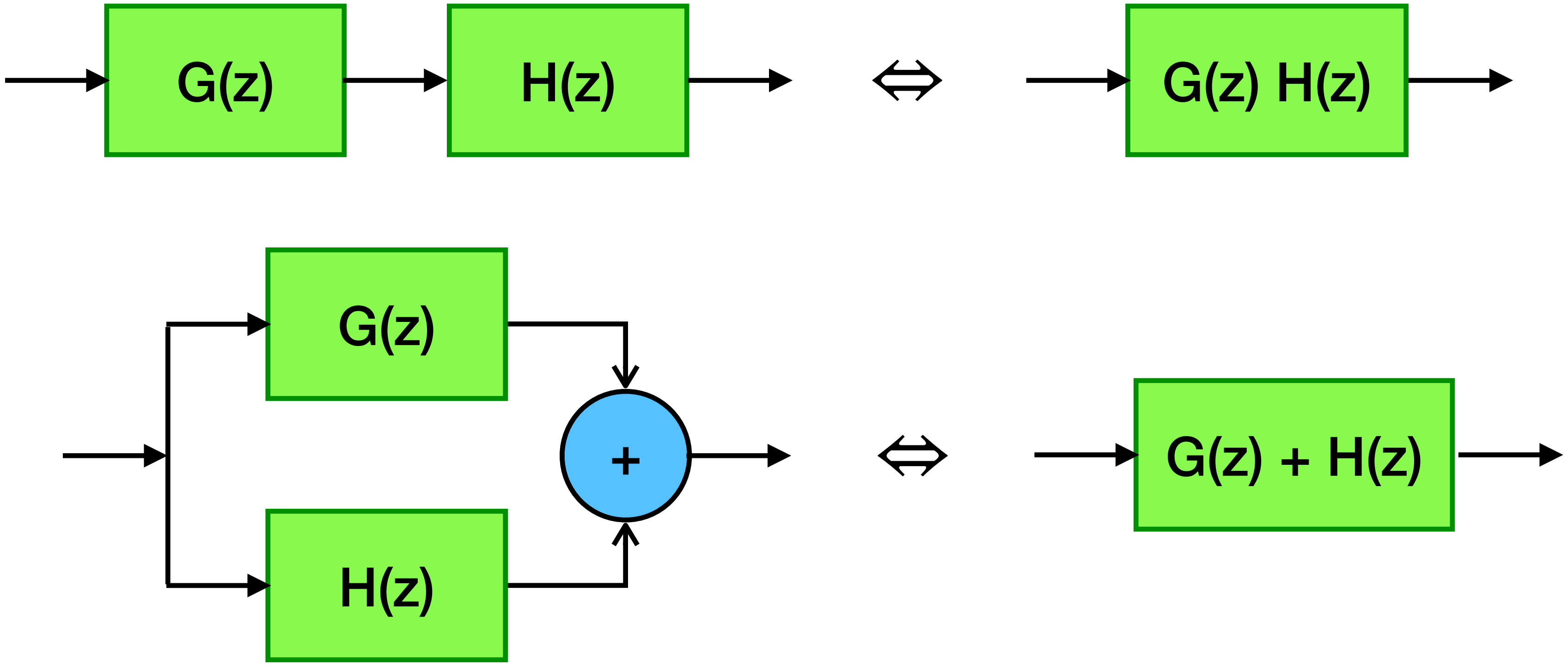
$$
\begin{aligned}
y[n] &= -1.91(0.5)^n u[n] \\
&\quad + 6.78e^{-j28.7^\circ} e^{j\frac{\pi n}{4}} u[n] \\
&\quad + 6.78e^{j28.7^\circ} e^{-j\frac{\pi n}{4}} u[n] \\
&= \left(-1.91(0.5)^n + 13.56\cos\left(\frac{\pi n}{4} - \frac{\pi 28.7^\circ}{180^\circ}\right)\right) u[n]
\end{aligned}
$$

Inverse Z-Transform Examples

See more examples in the textbook Chapter 3.

System Algebra

Series and parallel cascade equivalences.



If G(z)H(z) = 1, and H(z) = B(z)/A(z), then G(z) = A(z)/B(z).

$g[n] * h[n] = \delta[n]$

G(z) is the inverse system of H(z).

If H(z) is causal and stable, and G(z) is causal and stable, then all poles and zeros of H(z) must be inside the unit circle. This is called a minimum phase system.

System Algebra

$$H_1(z) = \frac{2 + 0.8z^{-1} - 2.2z^{-2}}{1 + 0.3z^{-1} - 0.4}$$

$$H_2(z) = \frac{-0.4z^{-1} + z^{-2}}{1 + 0.5z^{-1} - 0.24z^{-2}}$$

$$H(z) = H_1(z) + H_2(z) = \frac{2 - 1.8z^{-1} + 0.2z^{-2}}{1 - 0.8z^{-1} + 0.15z^{-2}}$$

```
n1=[2, 0.8, -2.2]; d1=[1, 0.3, -0.4];
n2=[0, -0.4, 1]; d2=[1, 0.5, -0.24];
n = conv(n1,d2) + conv(n2,d1);
d = conv(d1,d2); % common denominator
z = roots(n); % zeros = [ 0.7702, -0.8, -0.8, 0.1298]
p = roots(d); % poles = [-0.8   , -0.8,  0.5, 0.3   ]
n = poly(z([1,4]); % cancel zeros at -0.8
  = 2*[1, -0.9, 0.1]
d = poly(p([3,4]); % cancel poles at -0.8
  = [1, -0.8, 0.15]
```

System Algebra

$$H_1(z) = \frac{2 + 0.8z^{-1} - 2.2z^{-2}}{1 + 0.3z^{-1} - 0.4}$$

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```

The End

Next we will apply z-transform concepts to analyze systems.