System Analysis using the Z-Transform

ECE 3640 Discrete-Time Signals and Systems
Utah State University
Spring 2020

Characterization of LTI Systems

LTI systems are characterized by their impulse response and their transfer function.

$$y[n] = h[n] * x[n] \quad \overset{\mathrm{ZT}}{\longleftrightarrow} \quad Y(z) = H(z)X(z)$$

$$H(f) = H(z)\big|_{z=e^{j2\pi f}}$$

frequency response

transfer function

unit circle

$$H(f) = |H(f)|e^{j\angle H(f)}$$

magnitude response

phase response

$$Y(f) = H(f)X(f)$$

$$|Y(f)|e^{j\angle Y(f)} = |H(f)||X(f)|e^{j[\angle H(f) + \angle X(f)]}$$

magnitude response

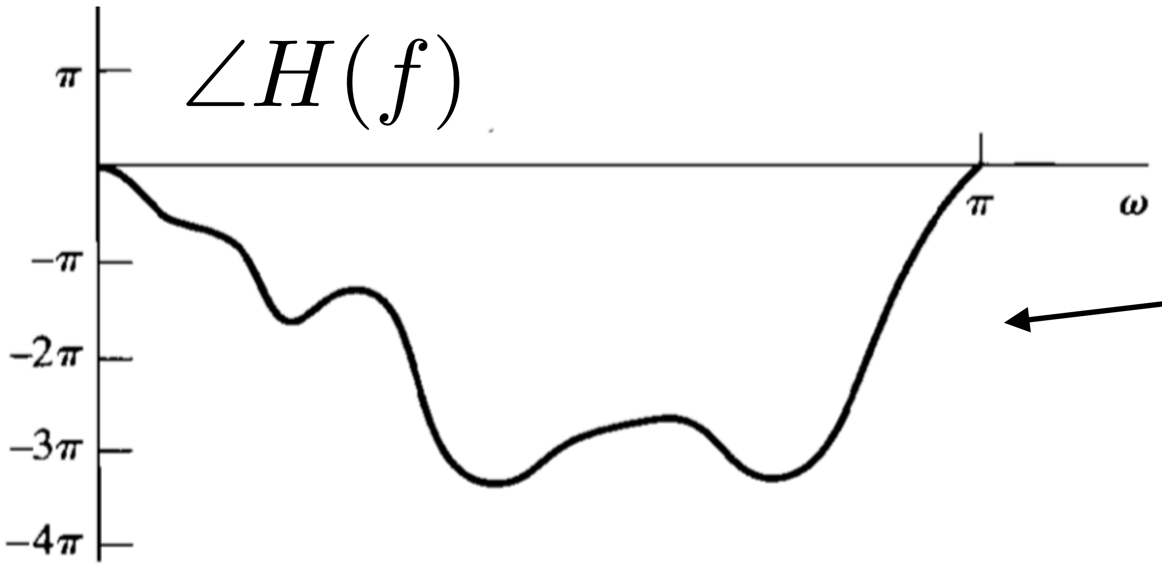
$$|Y(f)| = |H(f)|\,|X(f)|$$

$$|Y(f)|_{\mathrm{dB}} = |H(f)|_{\mathrm{dB}} + |X(f)|_{\mathrm{dB}}$$

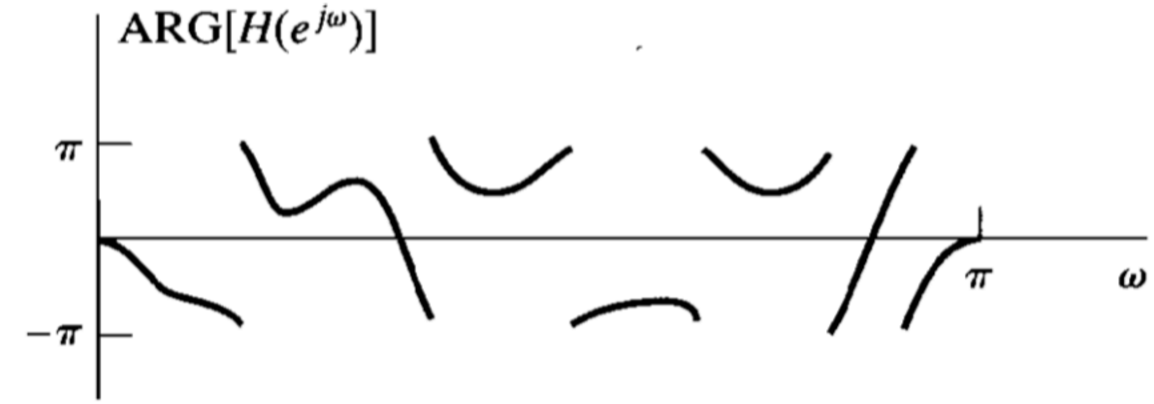
$$\angle Y(f) = \angle H(f) + \angle X(f)$$

phase response

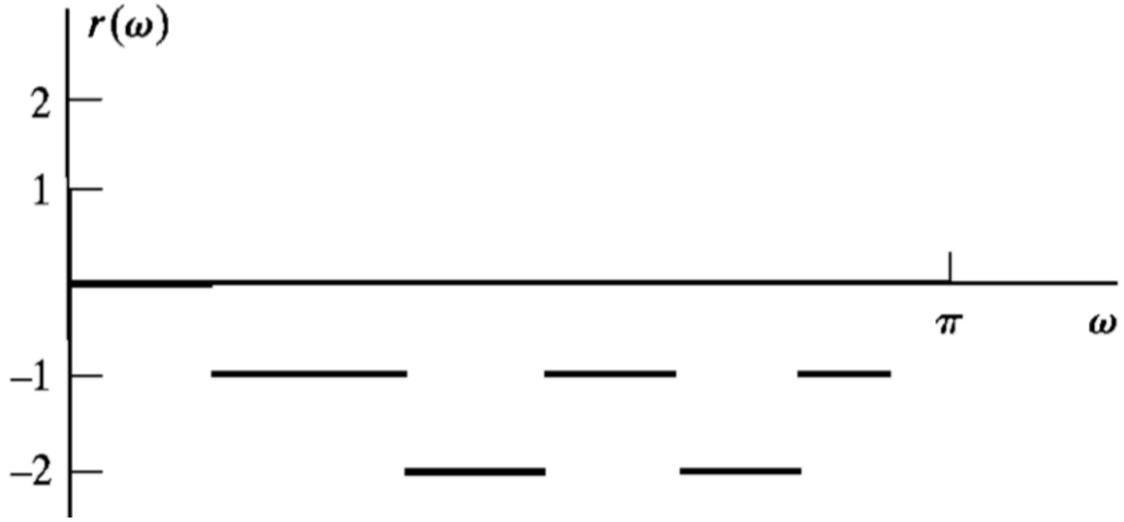
Phase Response



(a)



(b)



phase response
(unwrapped phase)
(continuous phase)

$$\angle H(f) = \mathrm{ARG}[H(f)] + 2\pi r(f)$$

principal value
(wrapped phase)

$$-\pi \leq \mathrm{ARG}[H(f)] \leq \pi$$

$$-\pi \leq \texttt{atan2}[H(f)] \leq \pi$$

$$r(f) \in \text{integers}$$

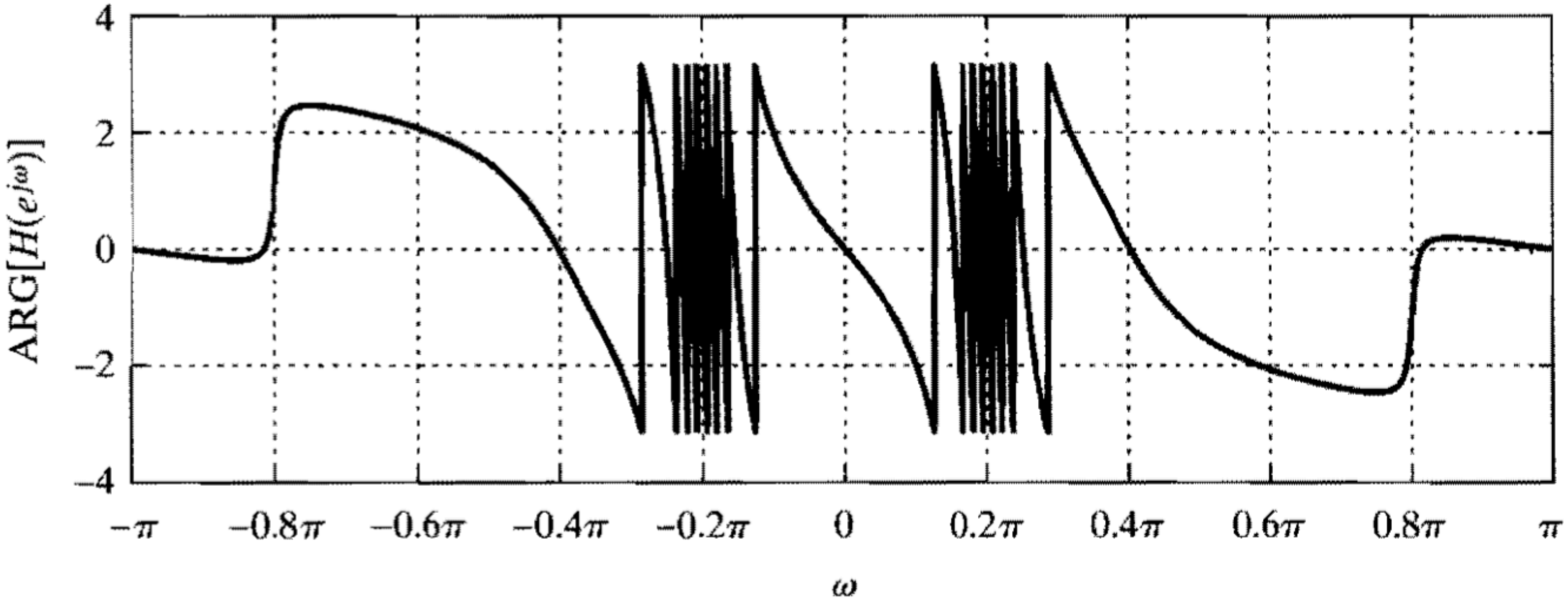
Phase Response

$$\angle H(f) = \mathrm{ARG}[H(f)] + 2\pi r(f)$$

$$\mathrm{ARG}[H(f)]$$

principal value
(wrapped phase)

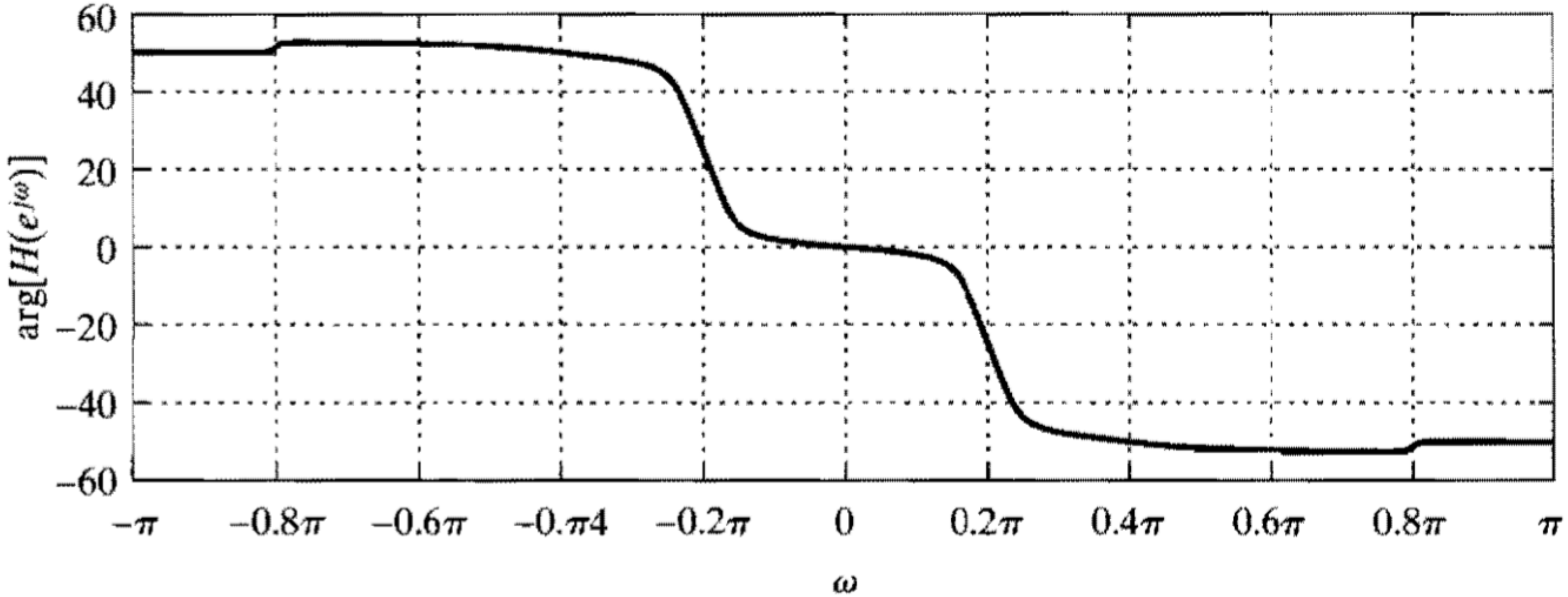
$$-\pi \leq \mathrm{ARG}[H(f)] \leq \pi$$



(a) Principle Value of Phase Response

$$\angle H(f)$$

phase response
(unwrapped phase)
(continuous phase)



(b) Unwrapped Phase Response

Group Delay

group delay is the slope of the phase response

$$\tau(f) = -\frac{1}{2\pi}\frac{d}{df}\angle H(f)$$

phase response
(unwrapped phase)
(continuous phase)

Example

$$h[n] = \delta[n-d]$$

$$H(f) = e^{-j2\pi f d}$$

$$\angle H(f) = -2\pi f d$$

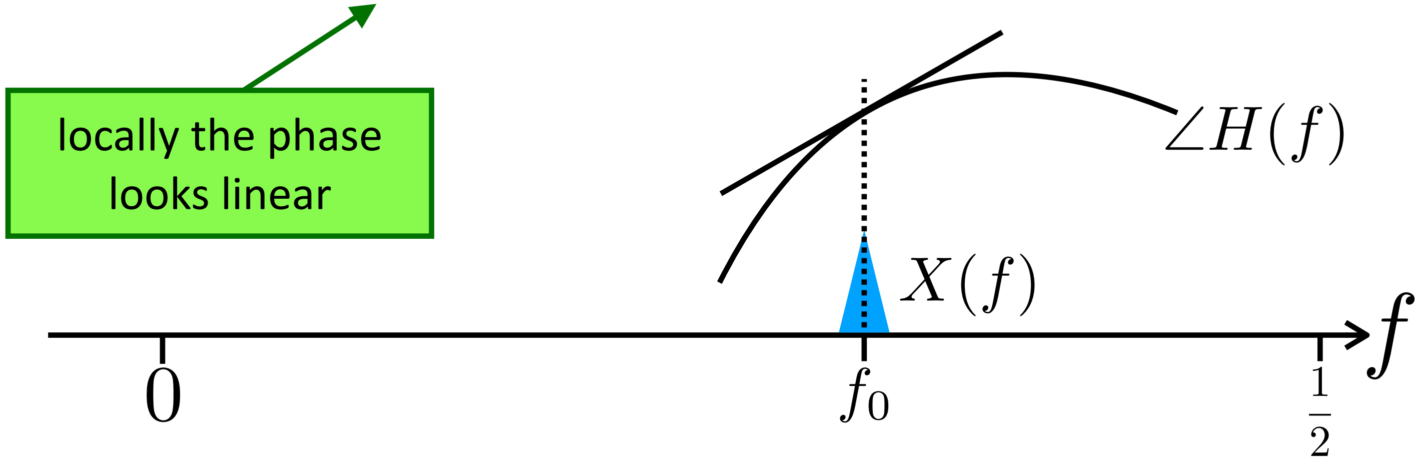
$$\tau(f) = d \text{ [samples]}$$

first-order Taylor approximation of $\angle H(f)$ about $f = f_0$

$$\angle H(f) \approx \angle H(f_0) - 2\pi(f - f_0)\tau(f_0)$$

$$= \varphi(f_0) - 2\pi f \tau(f_0)$$

locally the phase looks linear



$$x[n] = s[n]\cos(2\pi f_0 n)$$ (narrowband signal)

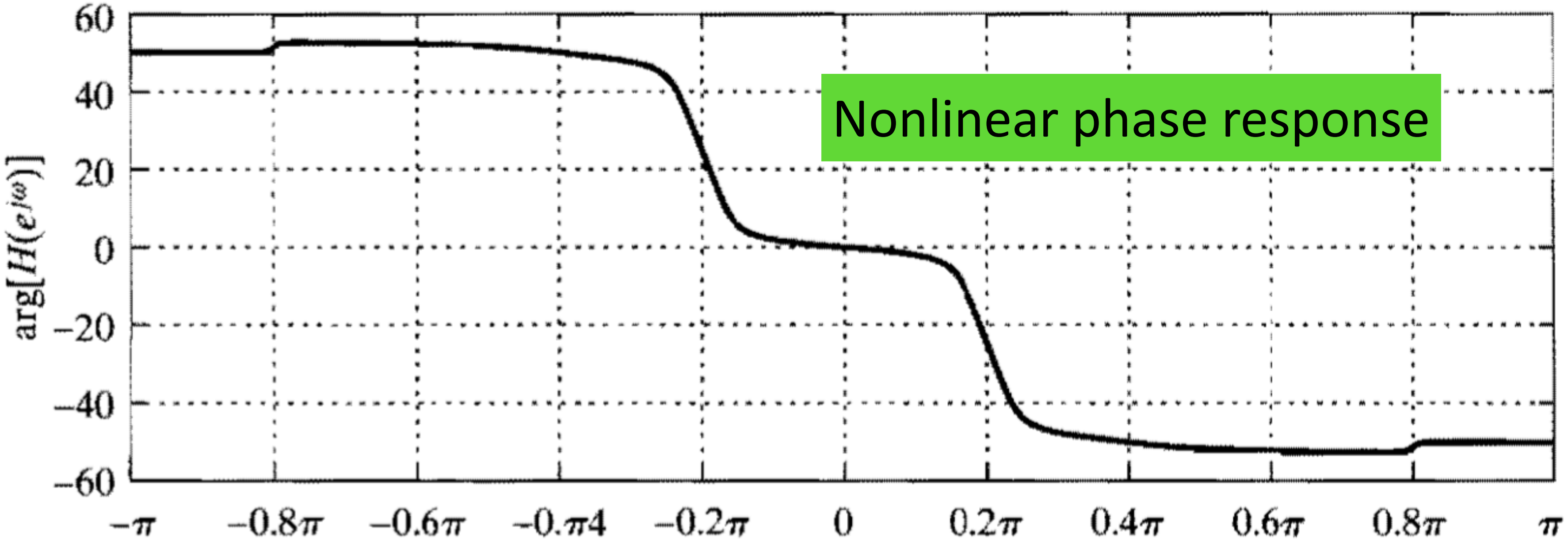
$$y[n] \approx |H(f_0)|s[n - \tau(f_0)]\cos(2\pi f_0 n + \varphi(f_0))$$

Broadband signals = sum of narrowband signals with different center frequencies.
Linear phase (constant group delay) ⇒ All components have same delay (distortionless system).
Nonlinear phase (variable group delay) ⇒ Different components have different delays (time dispersion).

Phase Response

$\angle H(f)$

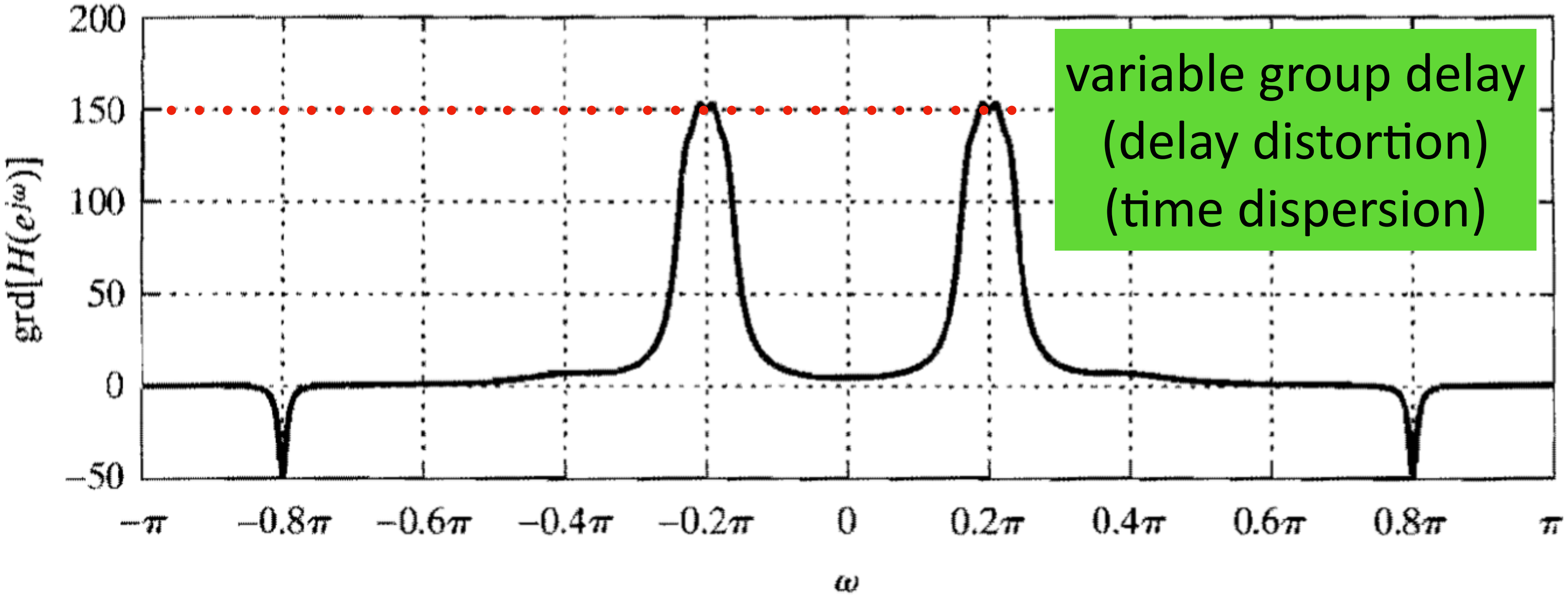
phase response
(unwrapped phase)
(continuous phase)



Nonlinear phase response

$\tau(f)$

group delay

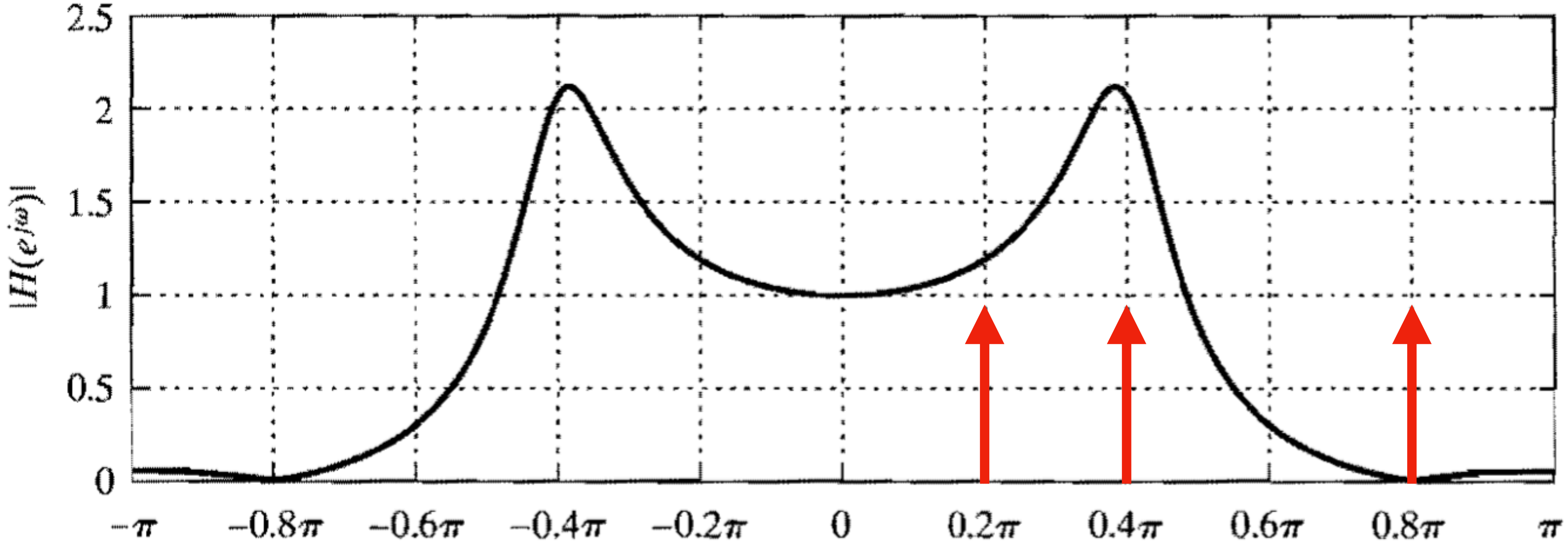


variable group delay
(delay distortion)
(time dispersion)

Phase Response Example

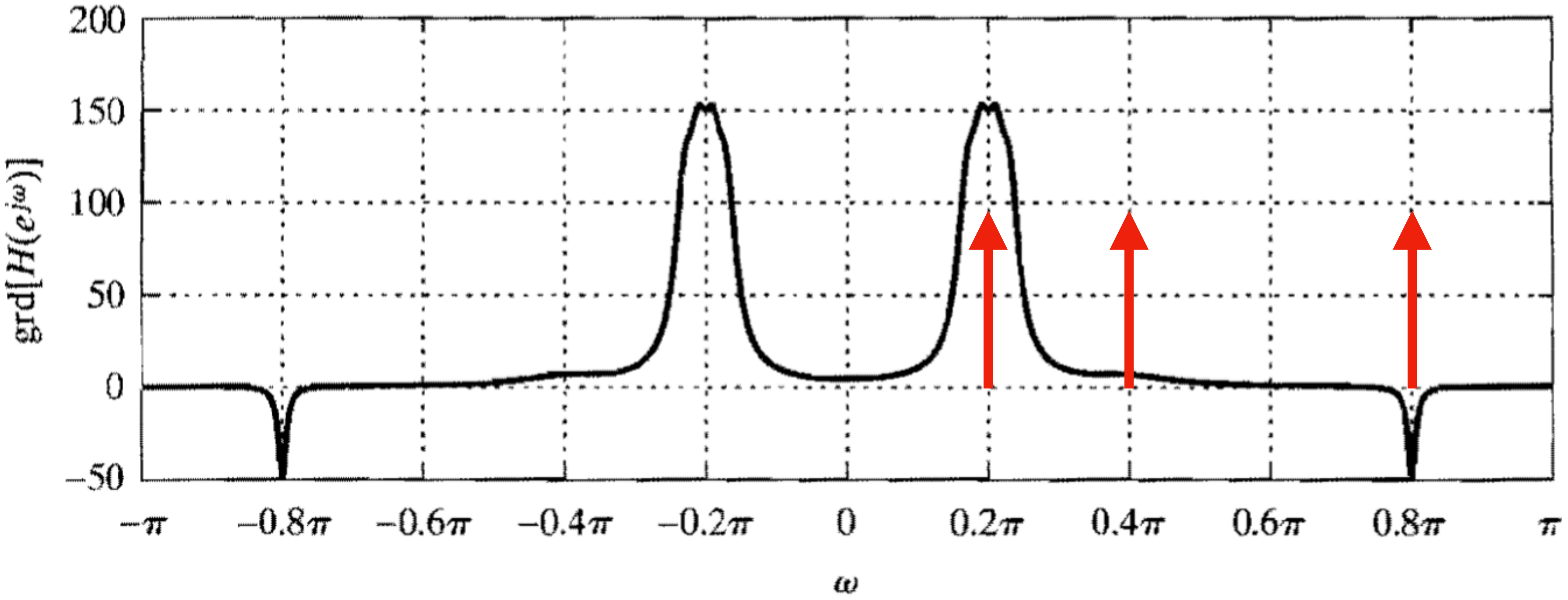
$|H(f)|$

magnitude response



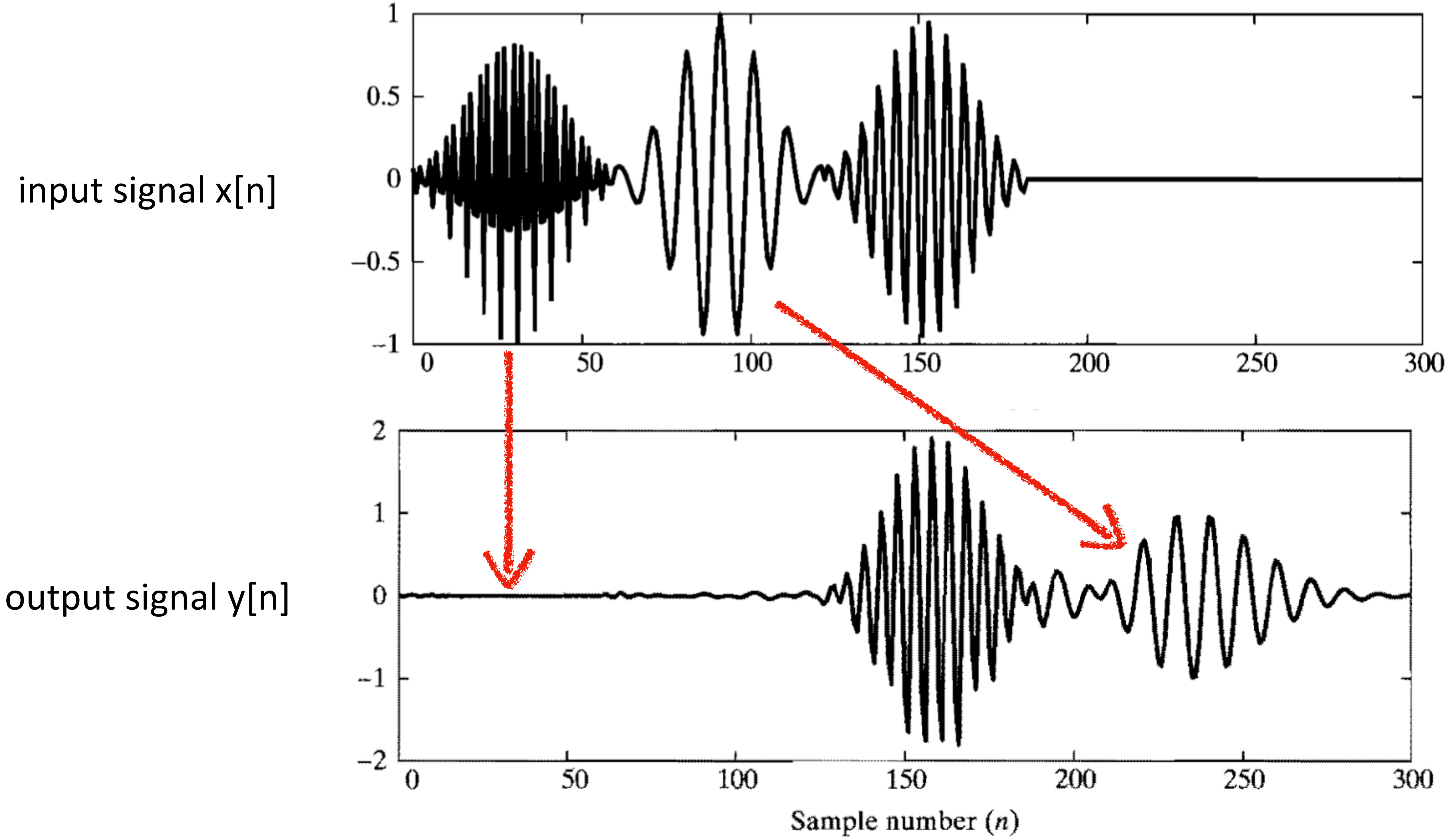
$\tau(f)$

group delay



The high frequency component is attenuated.
The mid frequency component has gain of two.
The low frequency component is delayed by 150 samples.

Phase Response Example



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The low frequency component is delayed by 150 samples.

Frequency Response for Rational Transfer Functions

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \Leftrightarrow \quad H(z)$$

$$H(z) = \frac{\displaystyle\sum_{k=0}^{M} b_k z^{-k}}{\displaystyle\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\displaystyle\prod_{k=1}^{M}(1 - z_k z^{-1})}{\displaystyle\prod_{k=1}^{N}(1 - p_k z^{-1})} \xrightarrow{z = e^{j2\pi f}} H(f) = \left(\frac{b_0}{a_0}\right) \frac{\displaystyle\prod_{k=1}^{M}(1 - z_k e^{-j2\pi f})}{\displaystyle\prod_{k=1}^{N}(1 - p_k e^{-j2\pi f})}$$

$$|H(f)| = \left|\frac{b_0}{a_0}\right| \frac{\displaystyle\prod_{k=1}^{M}|1 - z_k e^{-j2\pi f}|}{\displaystyle\prod_{k=1}^{N}|1 - p_k e^{-j2\pi f}|} \longleftarrow$$

magnitude response

$$|H(f)|_{\mathrm{dB}} = 20\log_{10}|H(f)|$$

$$|H(f)|_{\mathrm{dB}} = \left|\frac{b_0}{a_0}\right|_{\mathrm{dB}} + \sum_{k=1}^{M}|1 - z_k e^{-j2\pi f}|_{\mathrm{dB}} - \sum_{k=1}^{N}|1 - p_k e^{-j2\pi f}|_{\mathrm{dB}}$$

$$\angle H(f) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M}\angle(1 - z_k e^{-j2\pi f}) - \sum_{k=1}^{N}\angle(1 - p_k e^{-j2\pi f}) \longleftarrow$$

phase response

$$\tau(f) = -\sum_{k=1}^{M}\frac{|z_k|^2 - \mathrm{Re}\{z_k e^{-j2\pi f}\}}{1 + |z_k|^2 - 2\mathrm{Re}\{z_k e^{-j2\pi f}\}} + \sum_{k=1}^{N}\frac{|p_k|^2 - \mathrm{Re}\{p_k e^{-j2\pi f}\}}{1 + |p_k|^2 - 2\mathrm{Re}\{p_k e^{-j2\pi f}\}} \longleftarrow$$

group delay response

Frequency Response for Rational Transfer Functions

Single Zero

$$(1 - az^{-1}) = \frac{(z-a)}{z}$$

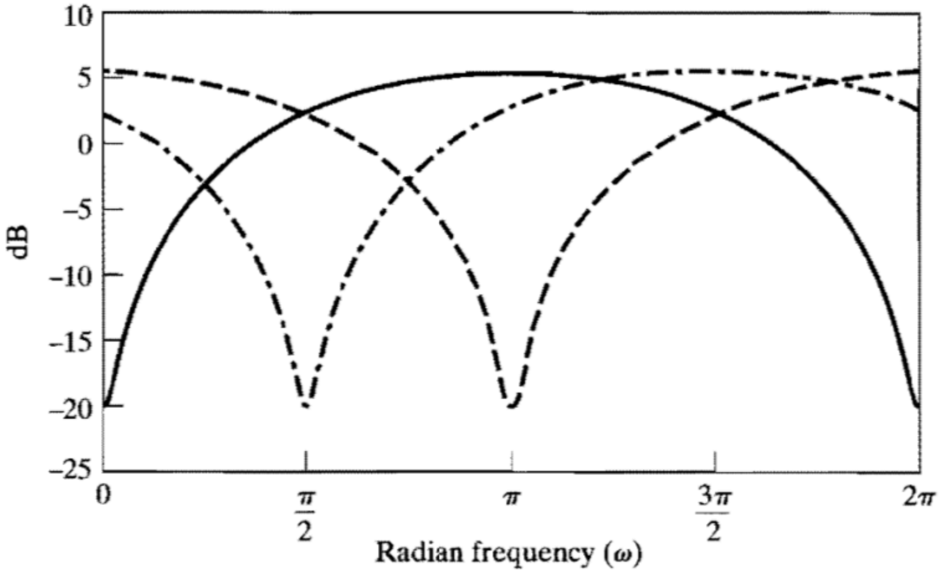
$$(1 - ae^{-j2\pi f})$$

$$\text{zero at } a = re^{j2\pi f}$$

$$r = 0.9, \quad f = 0, \frac{1}{4}, \frac{1}{2}$$

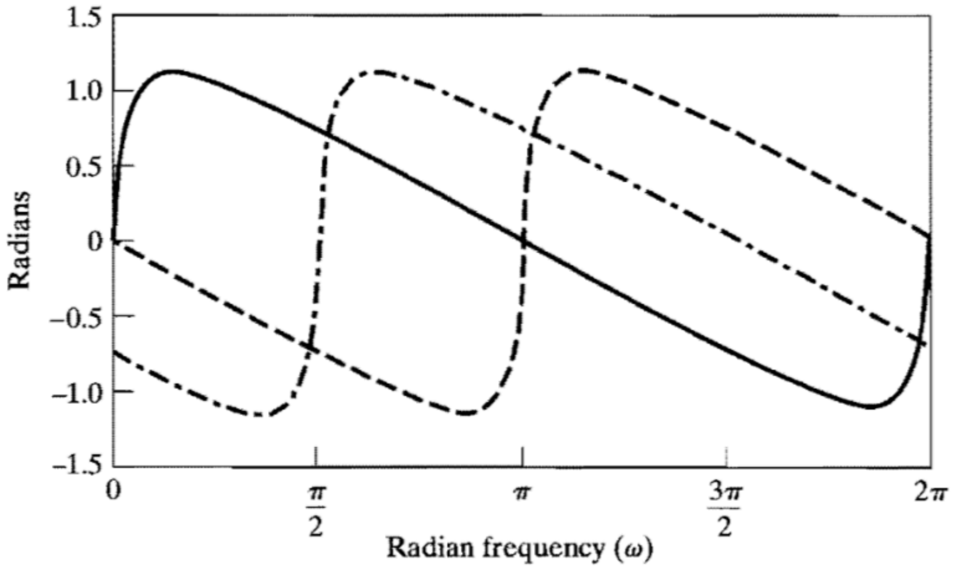
Single pole has the “opposite” response.

magnitude response



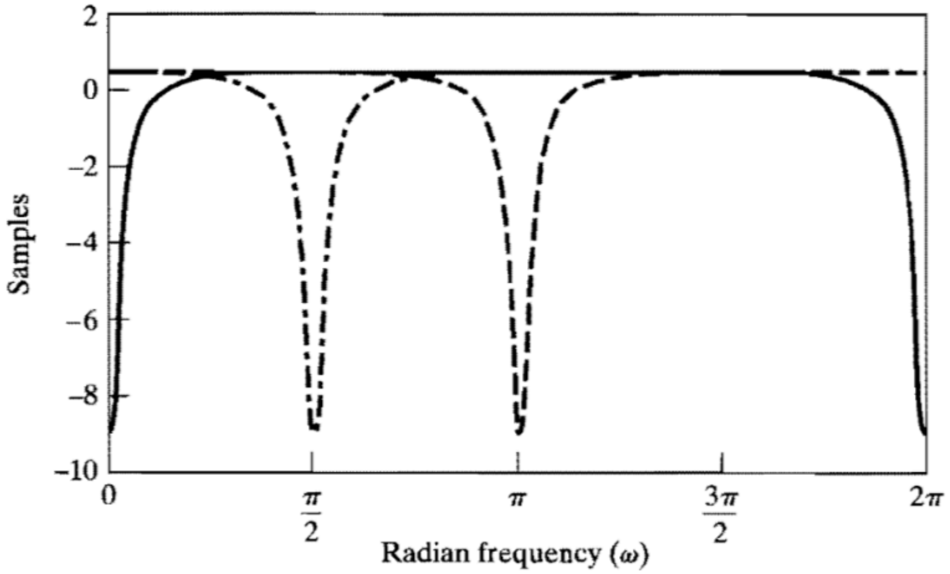
(a)

phase response



(b)

group delay



Frequency Response for Rational Transfer Functions

Single Zero

$$(1 - az^{-1}) = \frac{(z-a)}{z}$$

$$(1 - ae^{-j2\pi f})$$

$$\text{zero at } a = re^{j2\pi f}$$

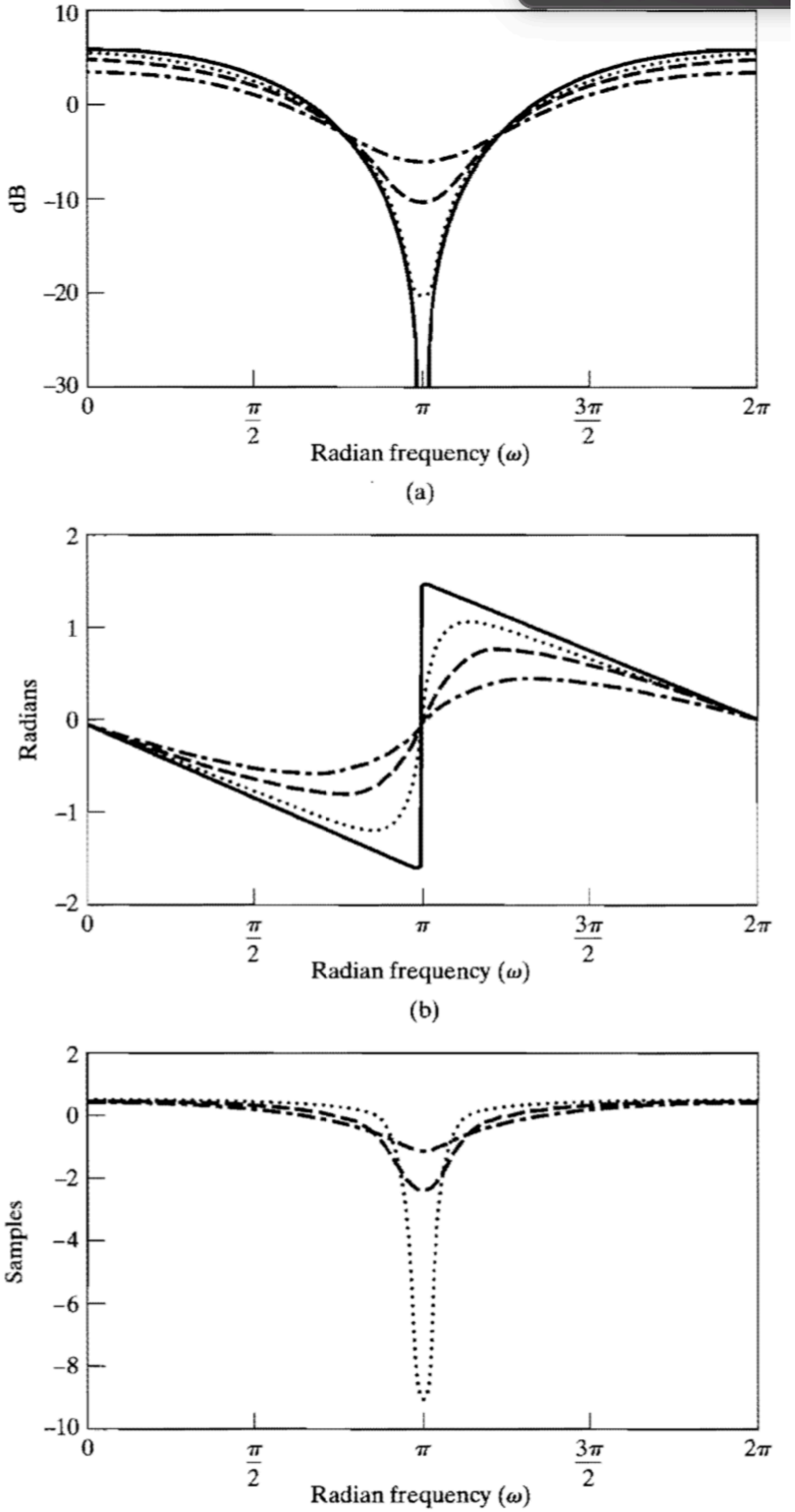
$$r = 0.5, 0.7, 0.9, 1.0, \quad f = \frac{1}{2}$$

Single pole has the "opposite" response.

magnitude response

phase response

group delay



Example 5.8 (page 299)

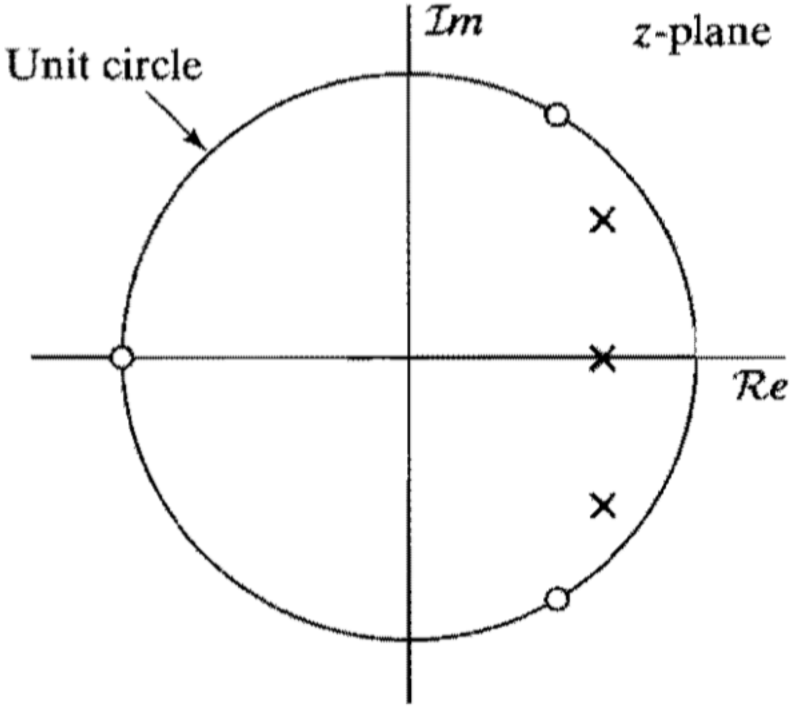
$$H(z) = \frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$$

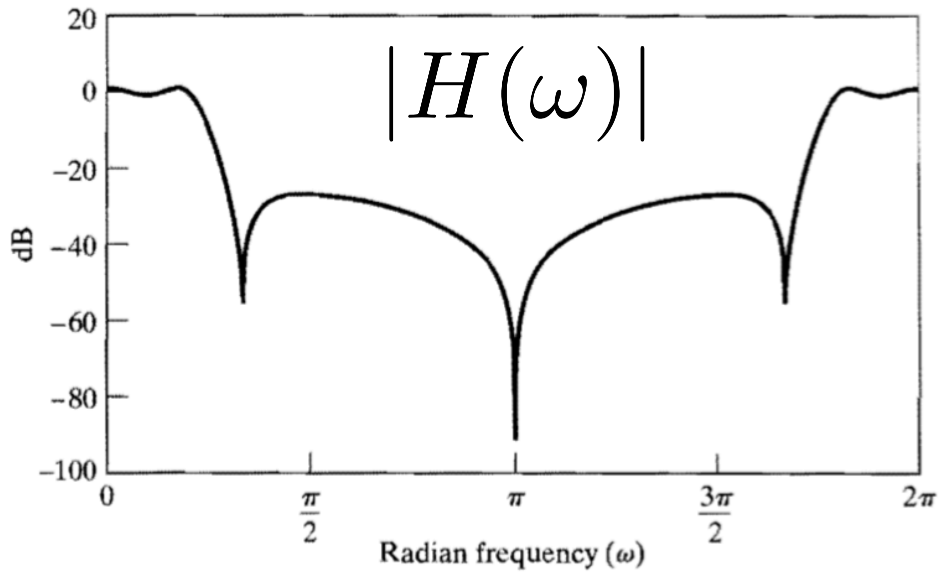
zeros

Radius	Angle
1	π rad
1	±1.0376 rad (59.45°)

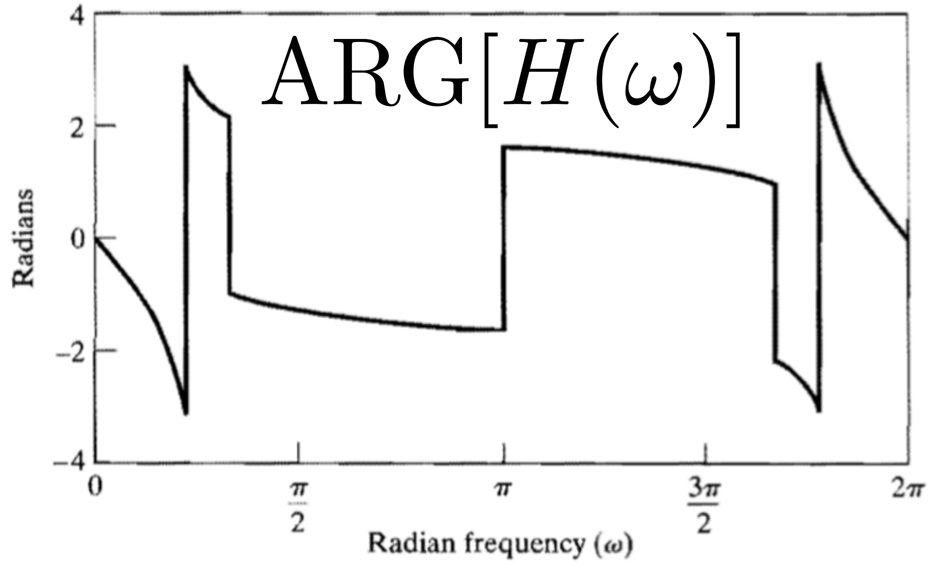
poles

Radius	Angle
0.683	0
0.892	±0.6257 rad (35.85°)

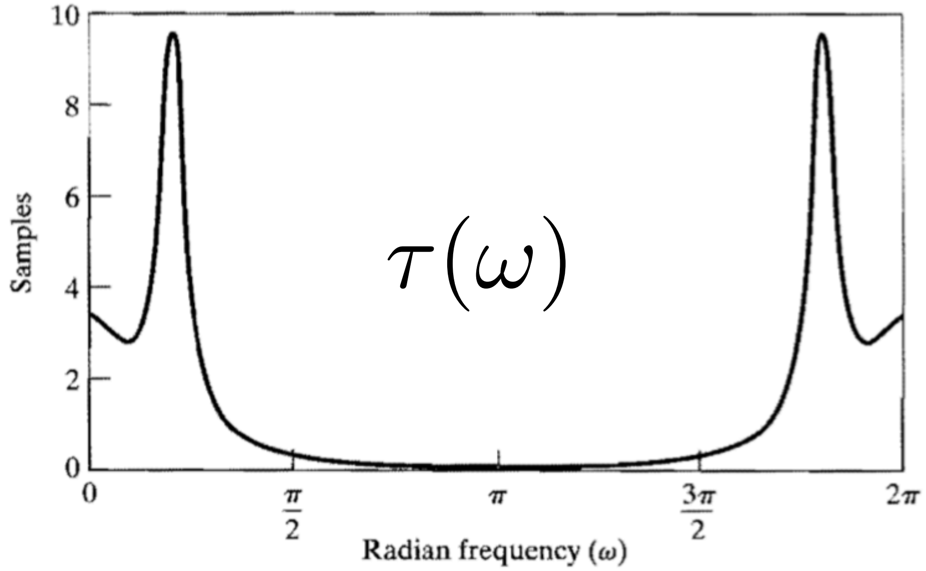




(a)



(b)



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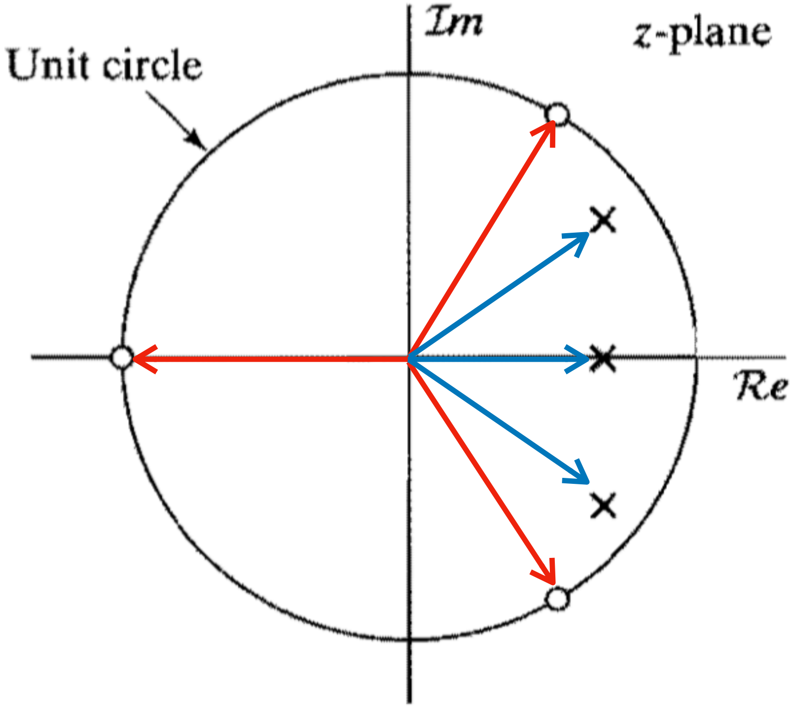
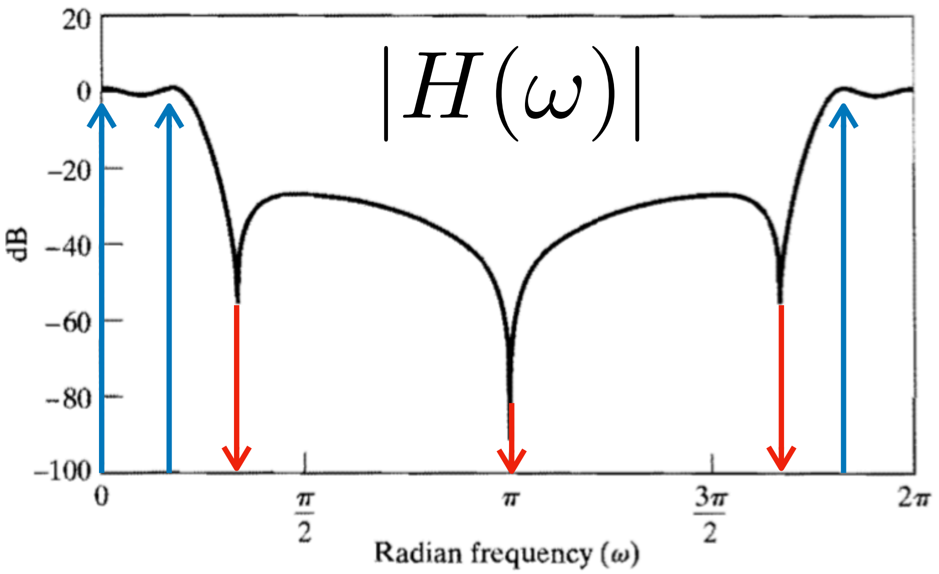
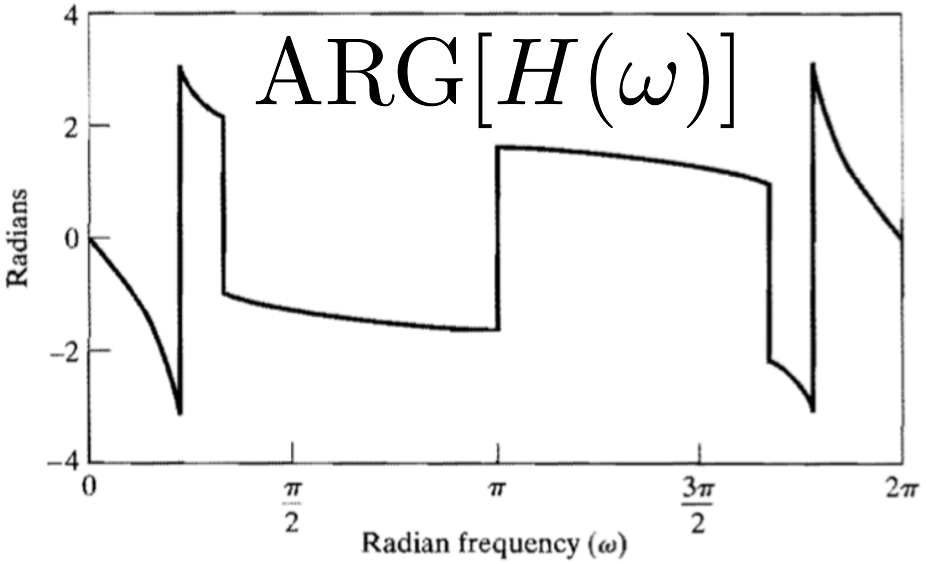




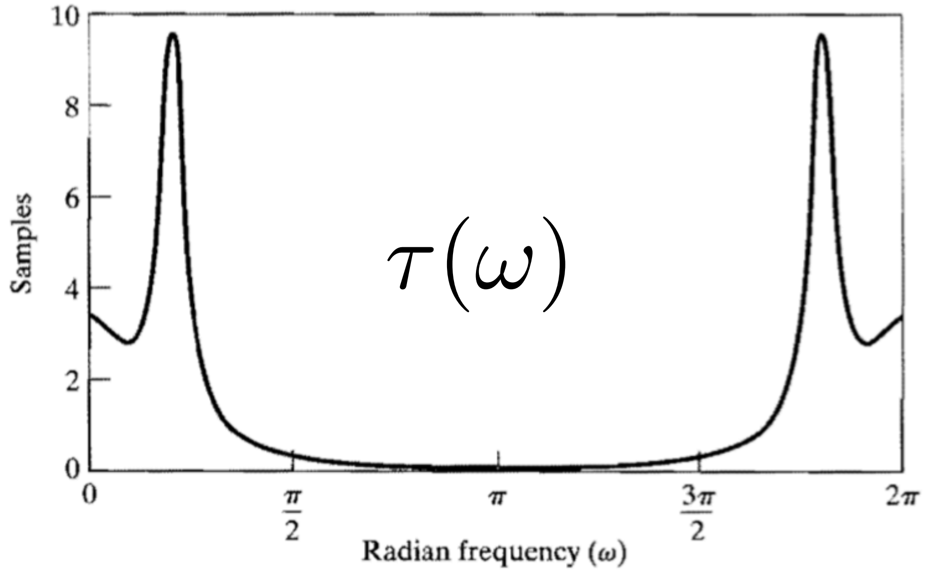
(a)

zeros pull the magnitude response down

poles push the magnitude response up



(b)

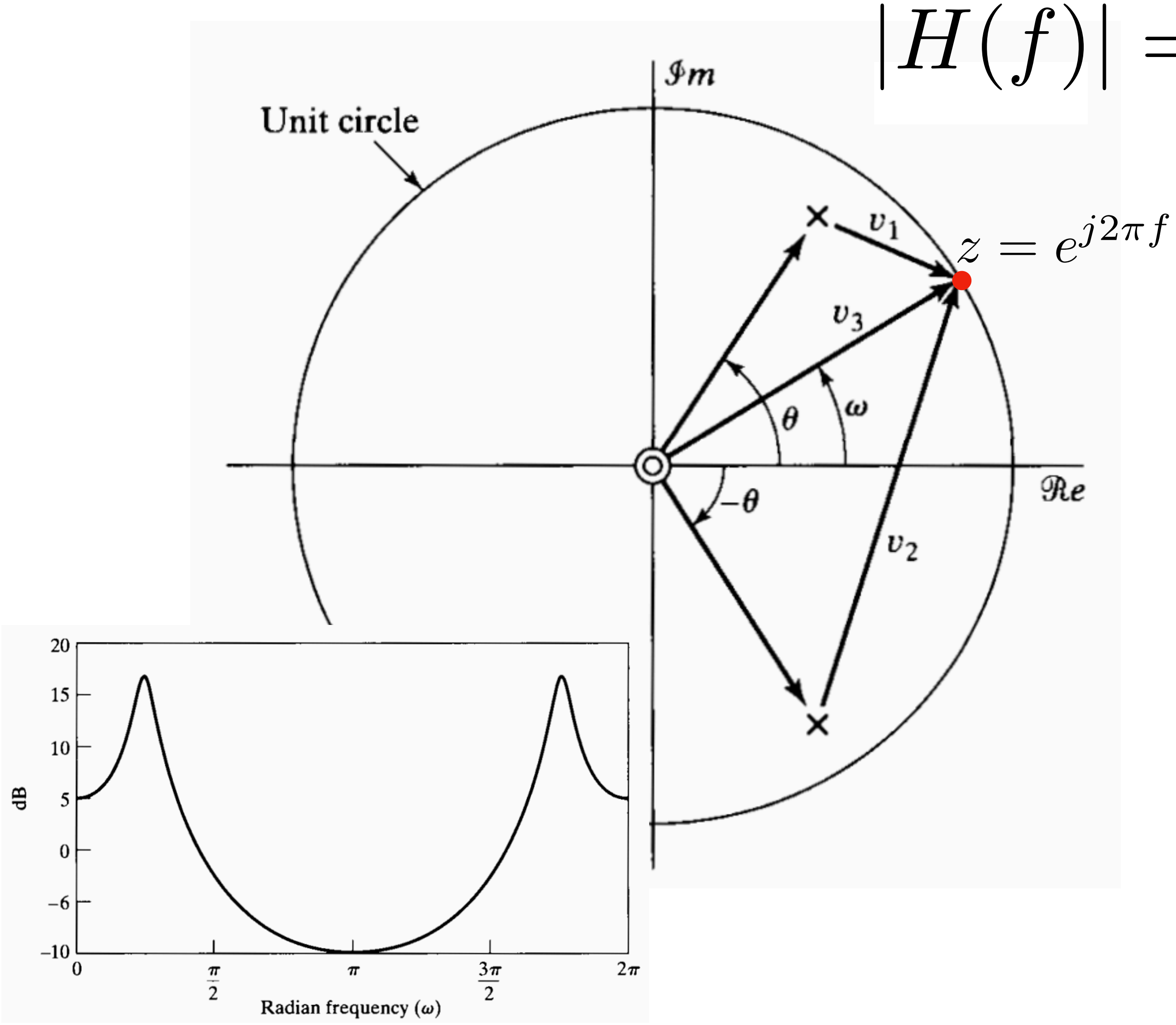


Magnitude Response Example

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n]$$

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{z^2}{(z - re^{j\theta})(z - re^{-j\theta})}$$

$$|H(f)| = \frac{|e^{j2\pi f}|^2}{|e^{j2\pi f} - re^{j\theta}| \cdot |e^{j2\pi f} - re^{-j\theta}|}$$



For any frequency and point $z = e^{j2\pi f}$, the magnitude response is the product of the lengths of “zero vectors” divided by the product of lengths of “pole vectors”.

$$|H(f)| = \frac{|v_3|^2}{|v_1| \cdot |v_2|} = \frac{1}{|v_1| \cdot |v_2|}$$

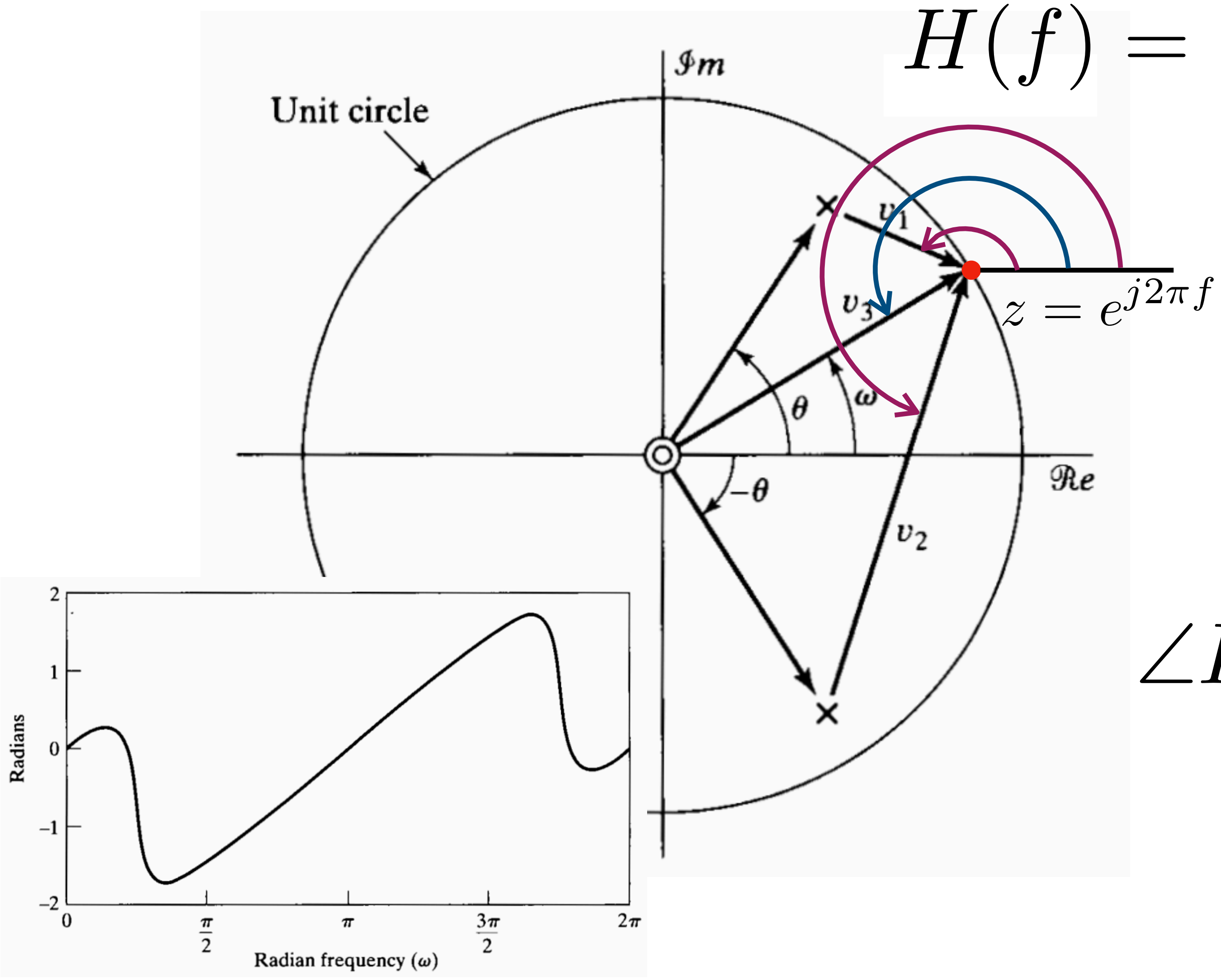
$$h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u[n]$$

Phase Response Example

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n]$$

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{z^2}{(z - re^{j\theta})(z - re^{-j\theta})}$$

$$H(f) = \frac{\left(e^{j2\pi f}\right)^2}{\left(e^{j2\pi f} - re^{j\theta}\right)\cdot\left(e^{j2\pi f} - re^{-j\theta}\right)}$$



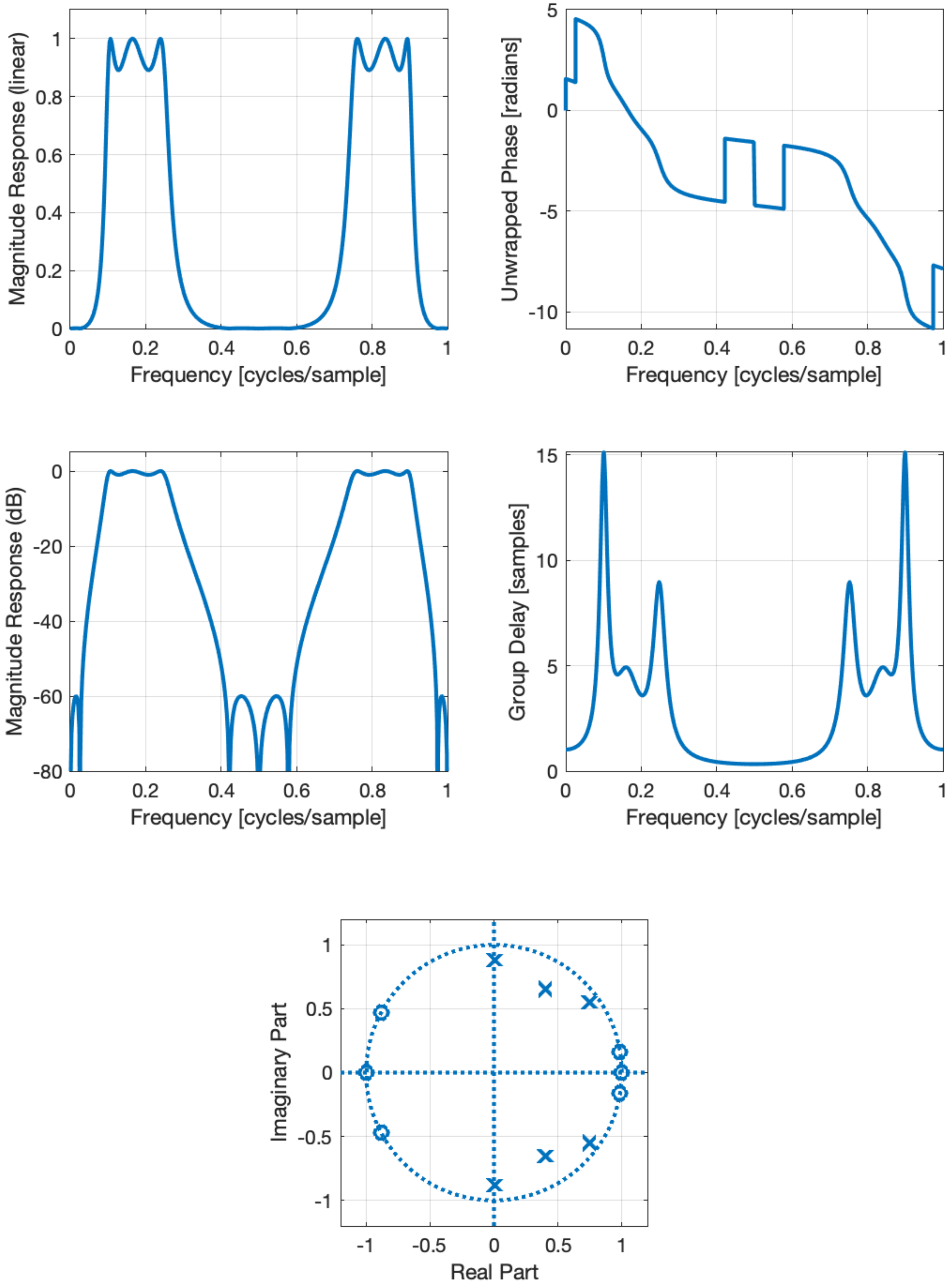
For any frequency and point $z = e^{j2\pi f}$, the phase response is the sum of the angles of “zero vectors” wrt the positive real axis minus the sum of angles of “pole vectors”.

$$\angle H(f) = 4\pi f - \angle\left(e^{j2\pi f} - re^{j\theta}\right) - \angle\left(e^{j2\pi f} - re^{-j\theta}\right)$$

$$h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u[n]$$

Band Pass Filter

Imagine combining responses of multiple zeros and poles to achieve magnitude, phase, and delay responses like those pictured here.



All-Pass Systems

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} = \frac{1 - a^* z}{z - a}$$

zero at a^{-*}

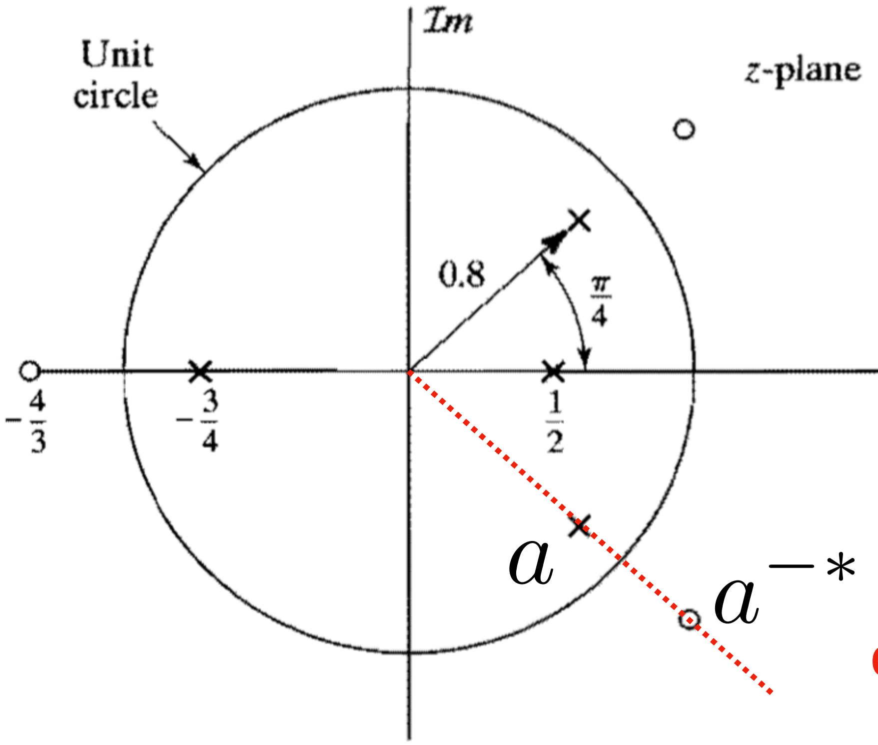
pole at a

$$H(f) = \frac{e^{-j2\pi f} - a^*}{1 - ae^{-j2\pi f}} = e^{-j2\pi f} \frac{1 - a^* e^{j2\pi f}}{1 - ae^{-j2\pi f}}$$

“all-pass”

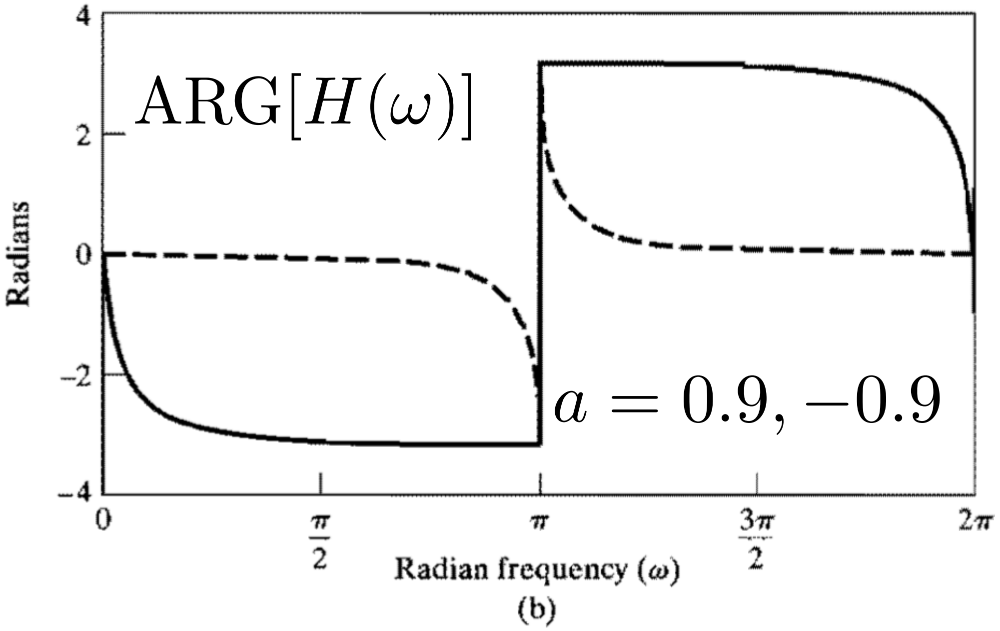
$$|H(f)| = 1$$

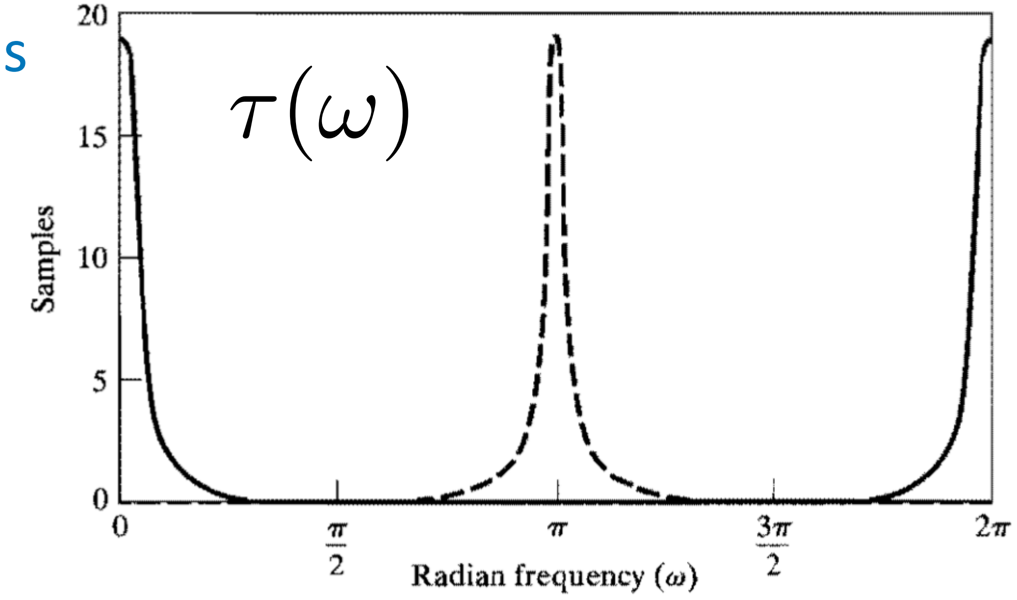
$$H(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - a_k}{1 - a_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - b_k^*)(z^{-1} - b_k)}{(1 - b_k z^{-1})(1 - b_k^* z^{-1})}$$



all-pass systems used to modify the phase response and group delay of other systems

poles and zeros at conjugate reciprocal locations fall on the same ray



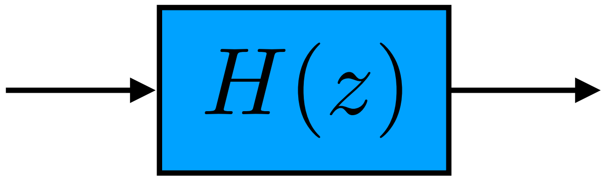


Minimum Phase Systems

1. All-pass (AP) systems have unity gain and pole-zero pairs at conjugate-reciprocal locations.
2. Minimum phase (MP) systems = All poles and zeros are inside the unit circle.
3. MP systems and their inverses are causal and stable.
4. General systems may be factored into MP and AP parts.
5. Can compensate for (or invert) the MP part of distortion.

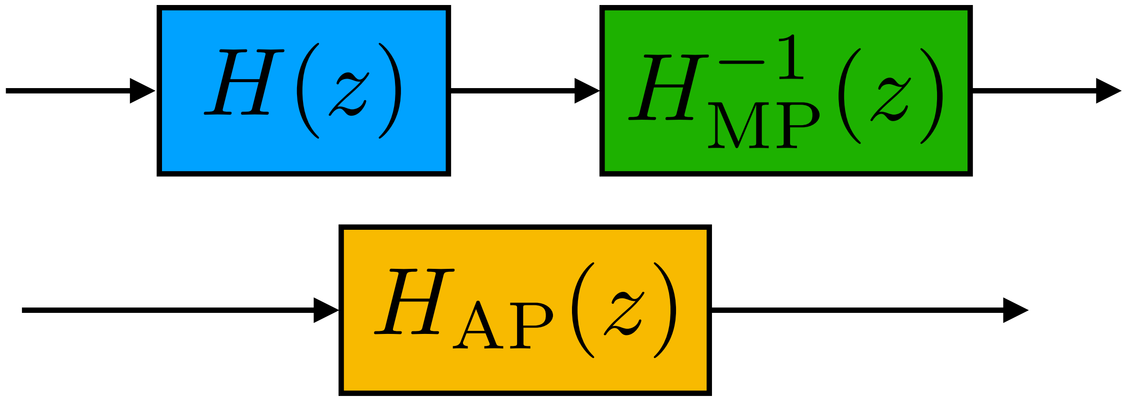
$$H(z) = H_{\mathrm{MP}}(z)H_{\mathrm{AP}}(z)$$

(decomposition)



$$H_{\mathrm{MP}}^{-1}(z)H(z) = H_{\mathrm{AP}}(z)$$

(compensation)



(no magnitude distortion, only phase distortion)

Minimum Phase Decomposition

How is the MP-AP decomposition computed?

Suppose $H(z)$ has one zero outside the unit circle at $z = c^{-*}$. Then $H(z)$ may be written as

$$\begin{aligned} H(z) &= H_1(z) \cdot (z^{-1} - c^*) \\ &= H_1(z) \cdot (1 - cz^{-1}) \cdot \left(\frac{z^{-1} - c^*}{1 - cz^{-1}} \right) \end{aligned}$$

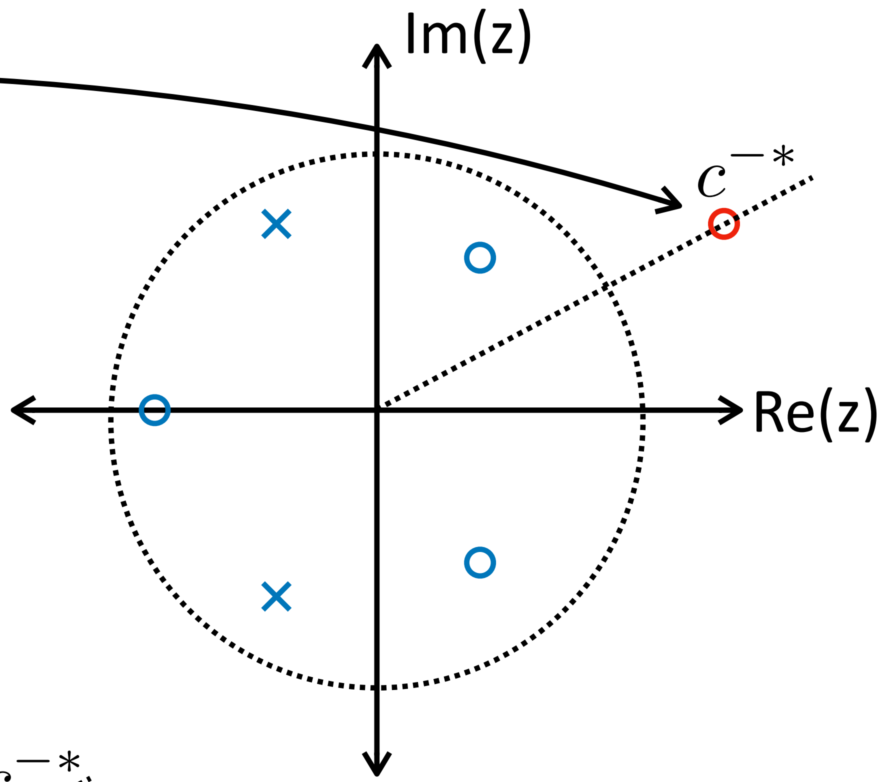
Minimum Phase Decomposition

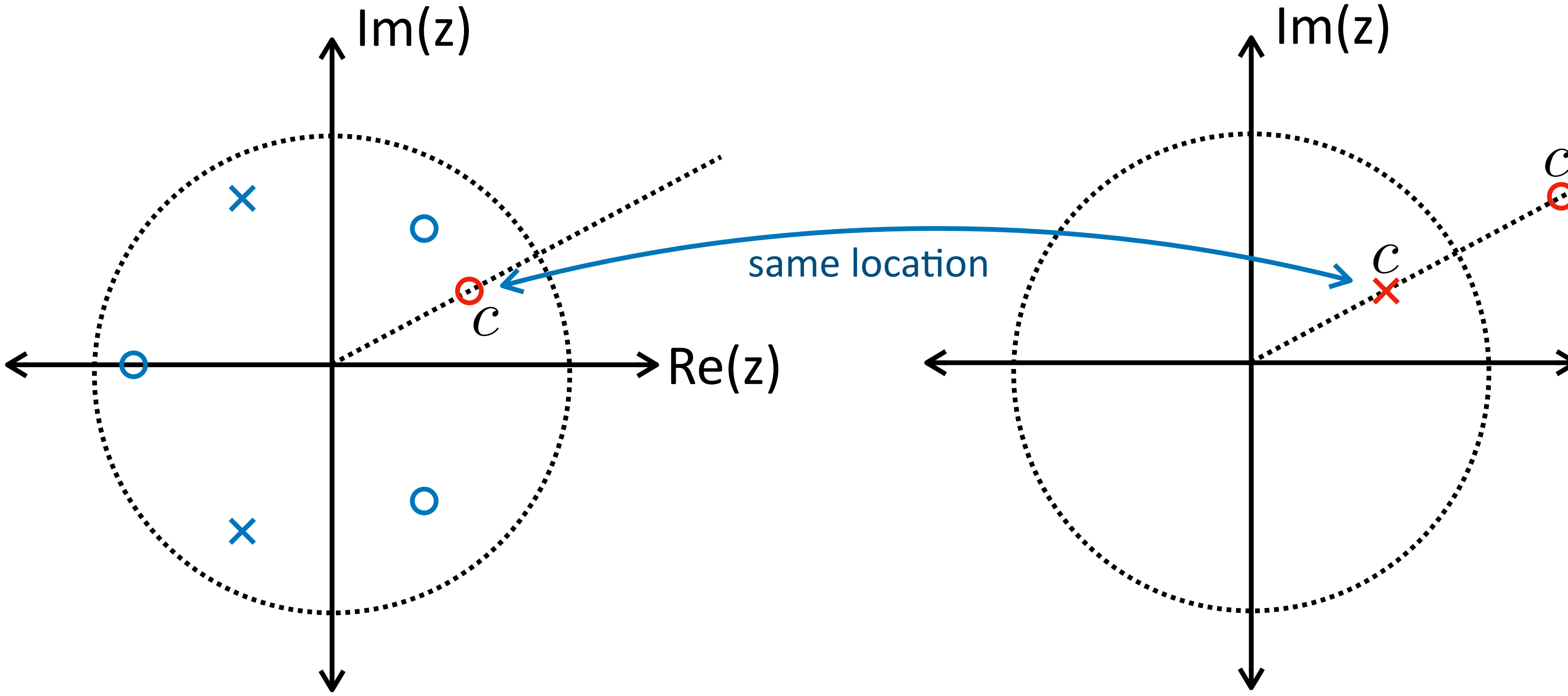
How is the MP-AP decomposition computed?

Suppose $H(z)$ has one zero outside the unit circle at $z = c^{-*}$. Then $H(z)$ may be written as

$$H(z) = H_1(z) \cdot (z^{-1} - c^*)$$

$$= \underbrace{H_1(z) \cdot (1 - cz^{-1})}_{\text{(minimum phase part)}} \cdot \underbrace{\left(\frac{z^{-1} - c^*}{1 - cz^{-1}} \right)}_{\text{(all-pass part)}}$$





The same process can be applied for poles outside the unit circle.

Matlab Examples

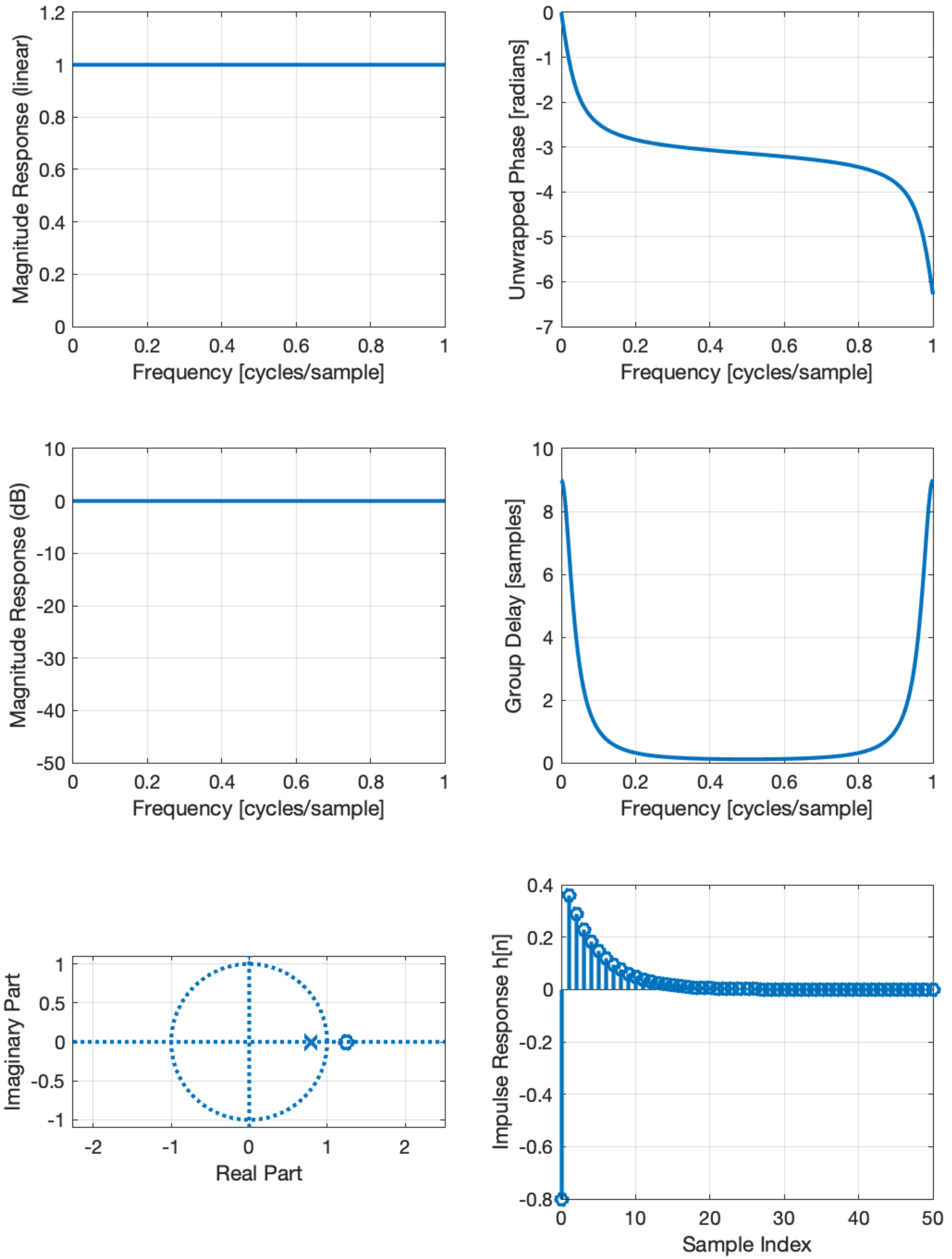
Examples in Matlab:

- All-pass, single pole
- All-pass, complex conjugate pole pair
- DC blocker
- FIR low pass filter
- IIR low pass filter

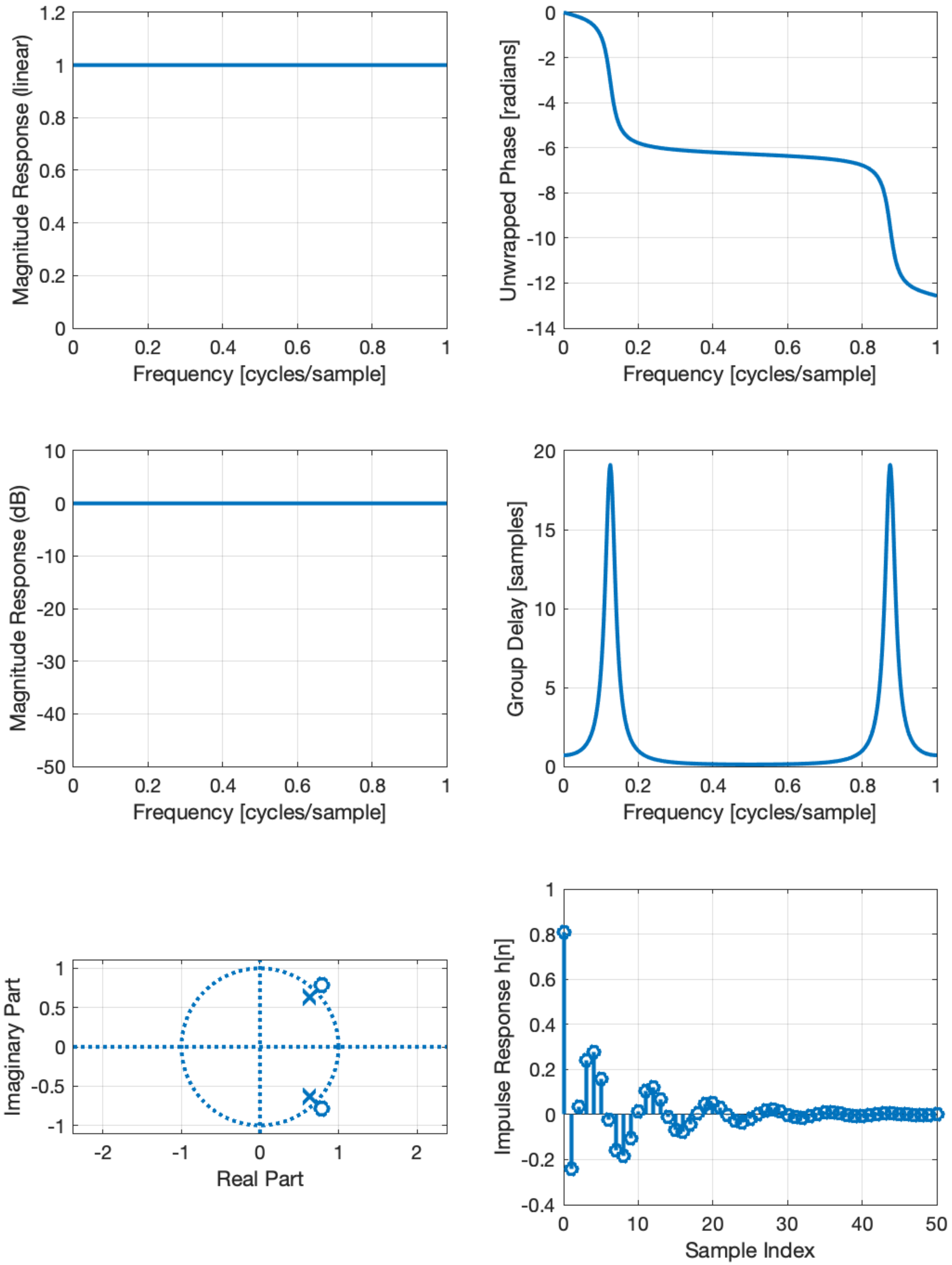
Look at:

- Magnitude response
- Phase response
- Group delay
- Pole-zero plot
- Impulse response

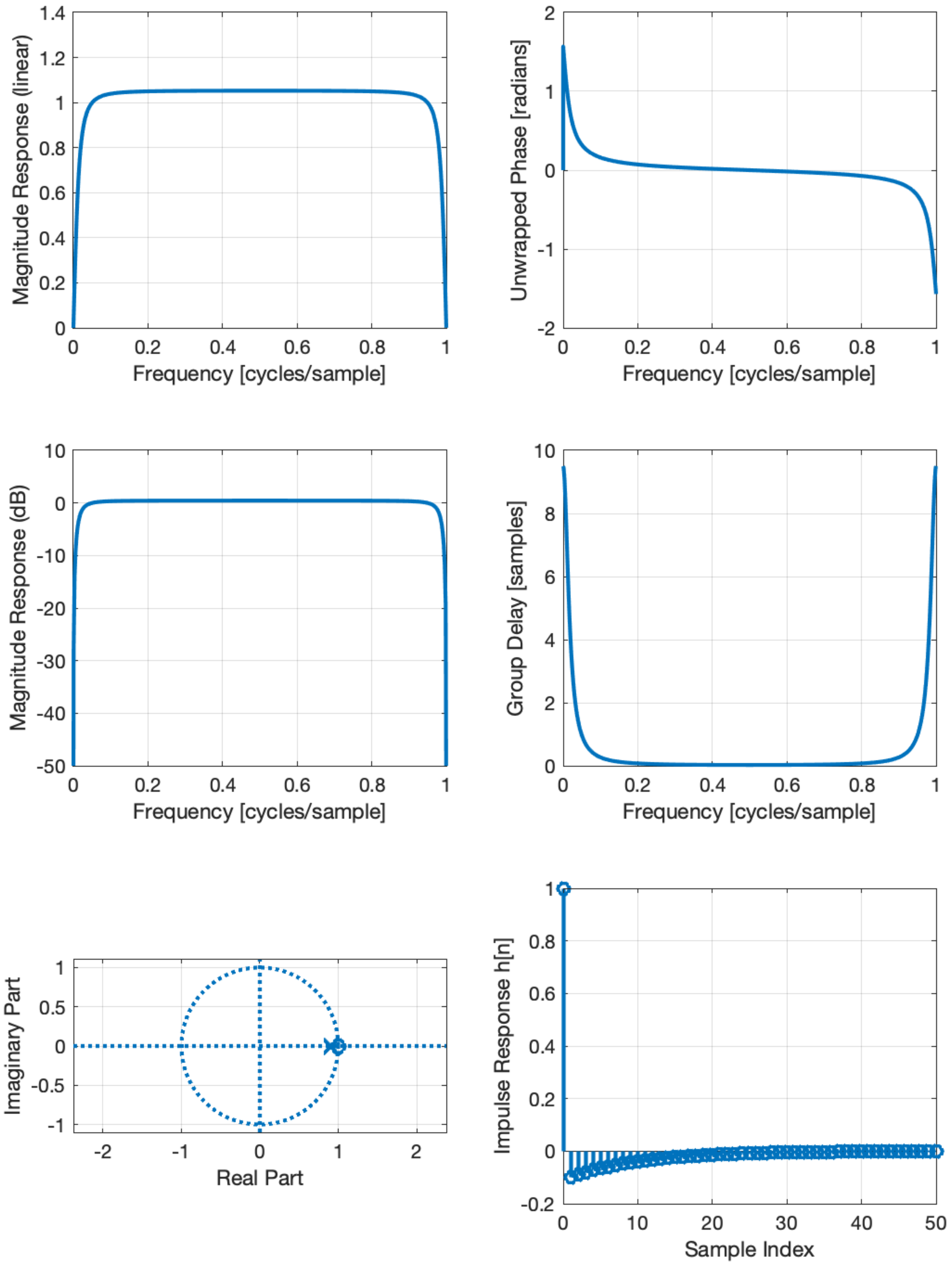
All-Pass Filter
Single Real Pole



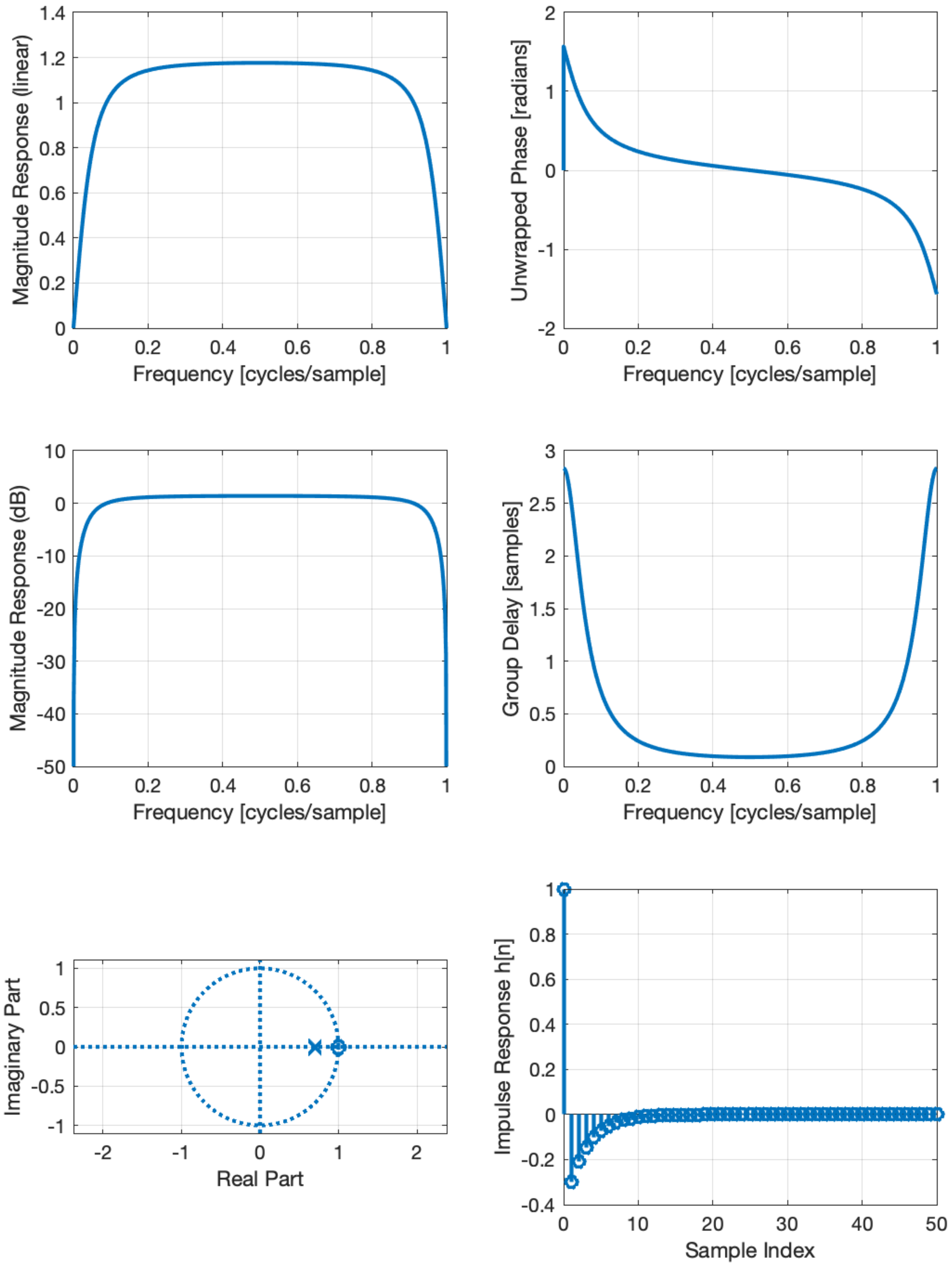
All-Pass Filter
Complex-Conjugate Poles



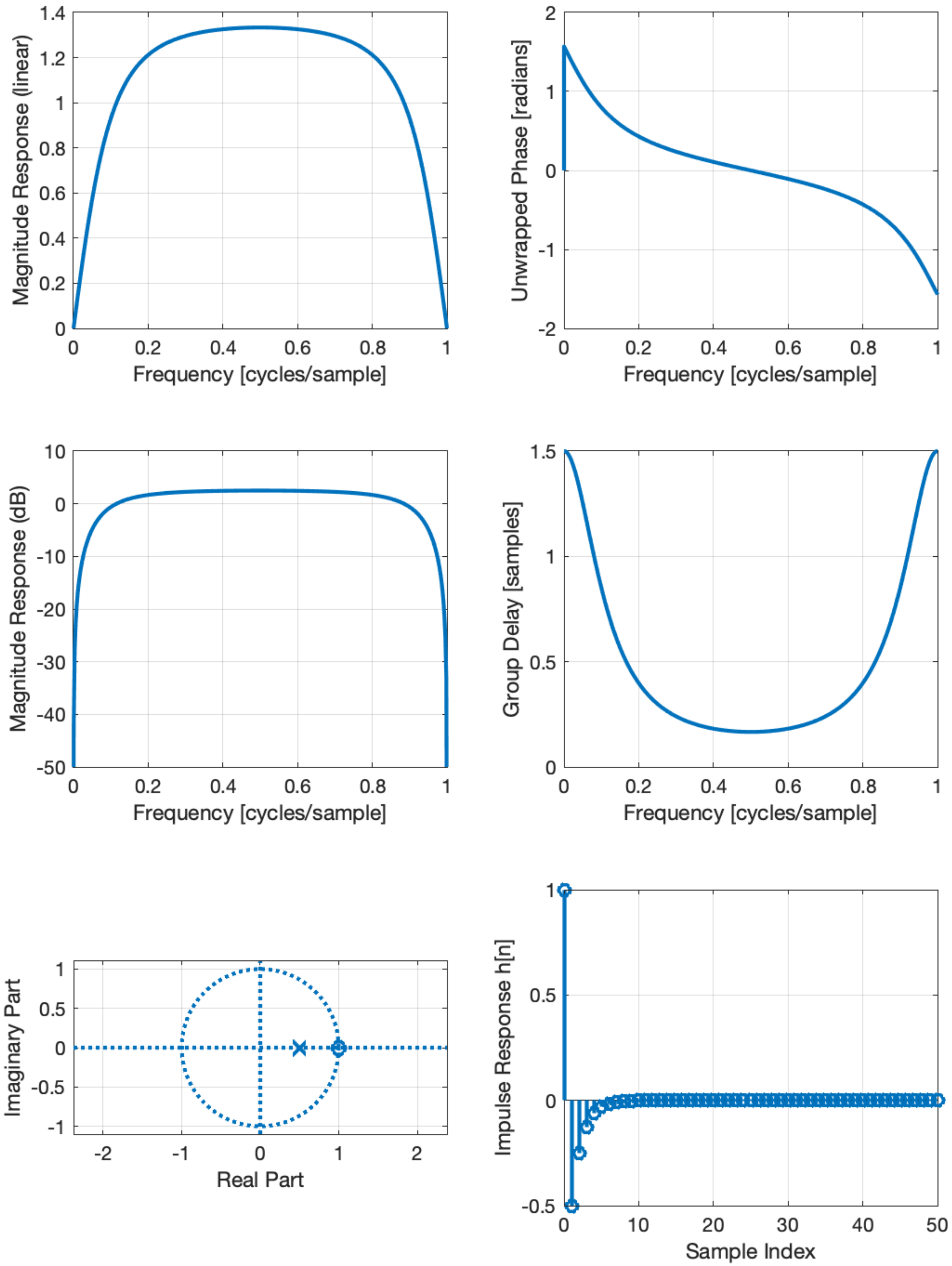
DC Blocker
Pole at z=0.9



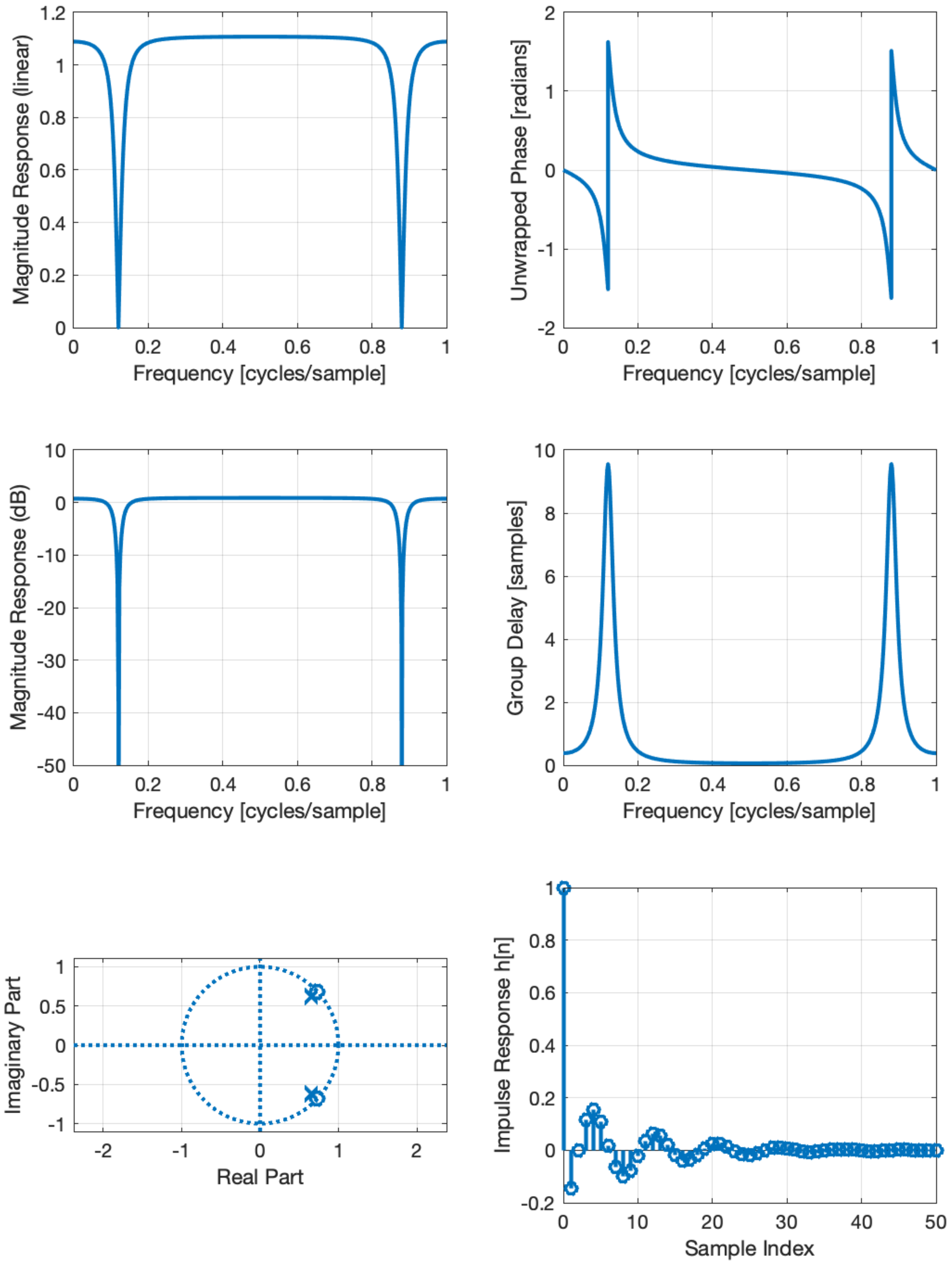
DC Blocker
Pole at z=0.7



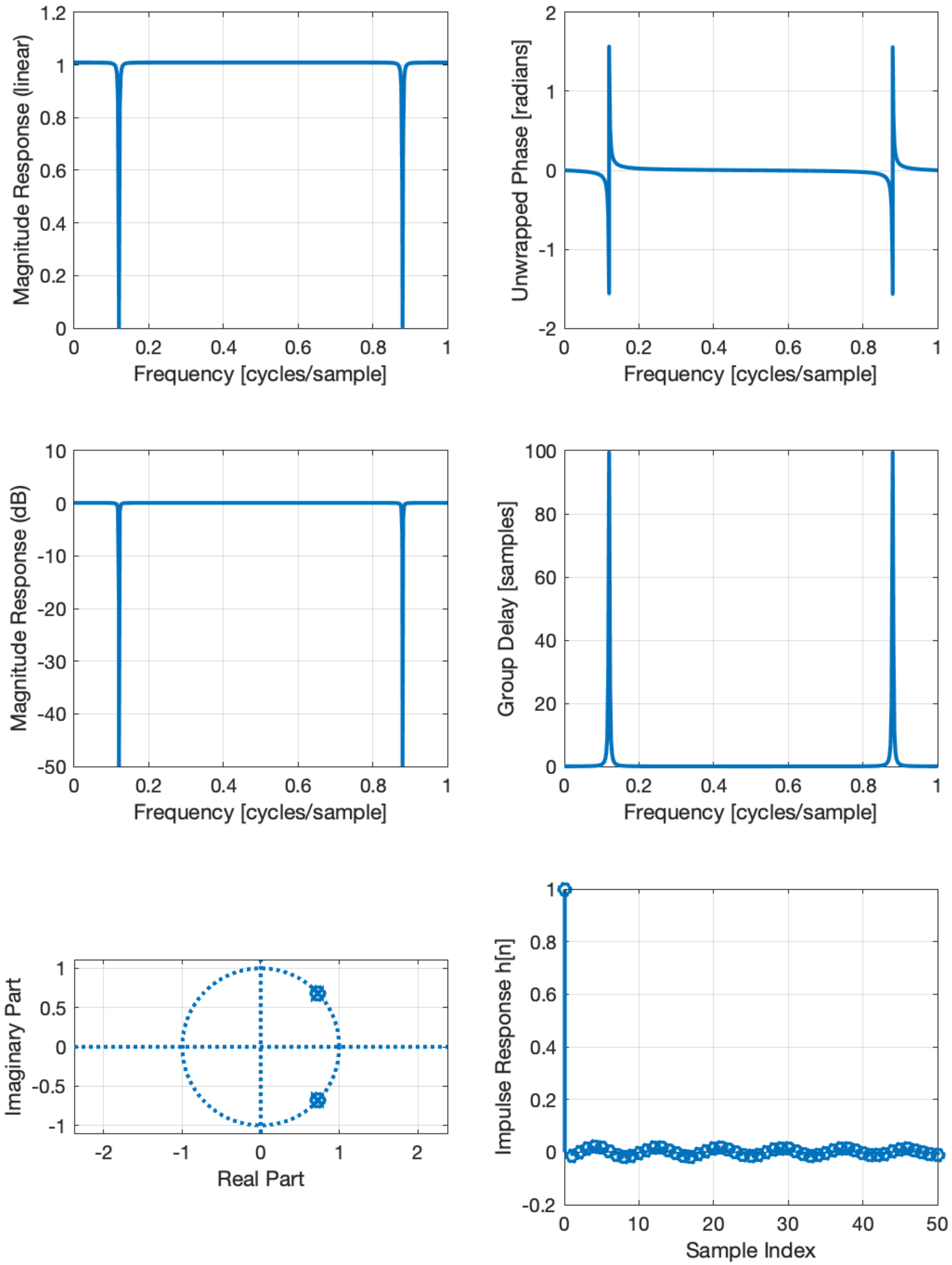
DC Blocker
Pole at z=0.5



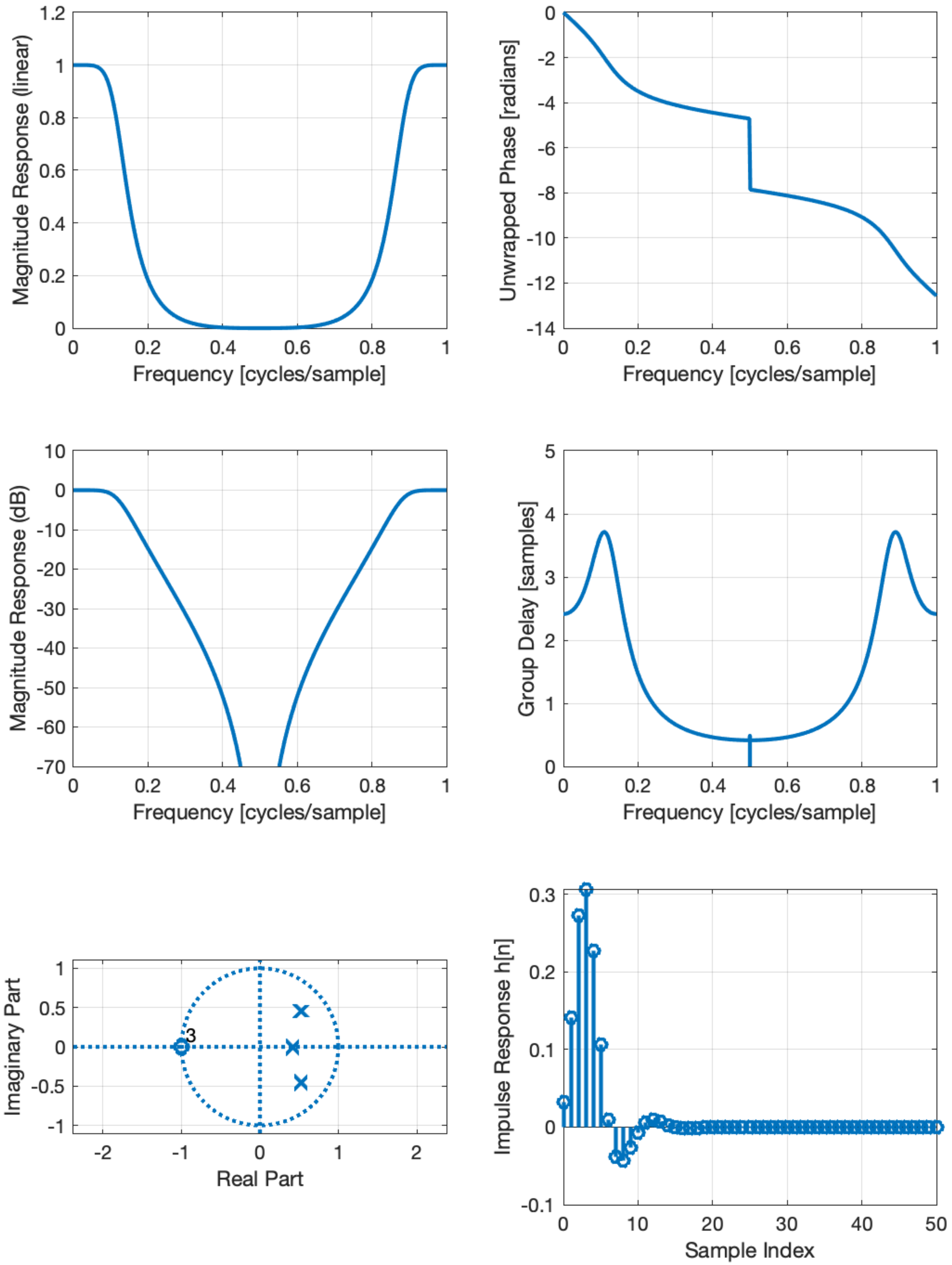
# 60 Hz Notch
Fs = 500 samples/second
Pole radius = 0.9
Pole angle = 2pi * 60/500



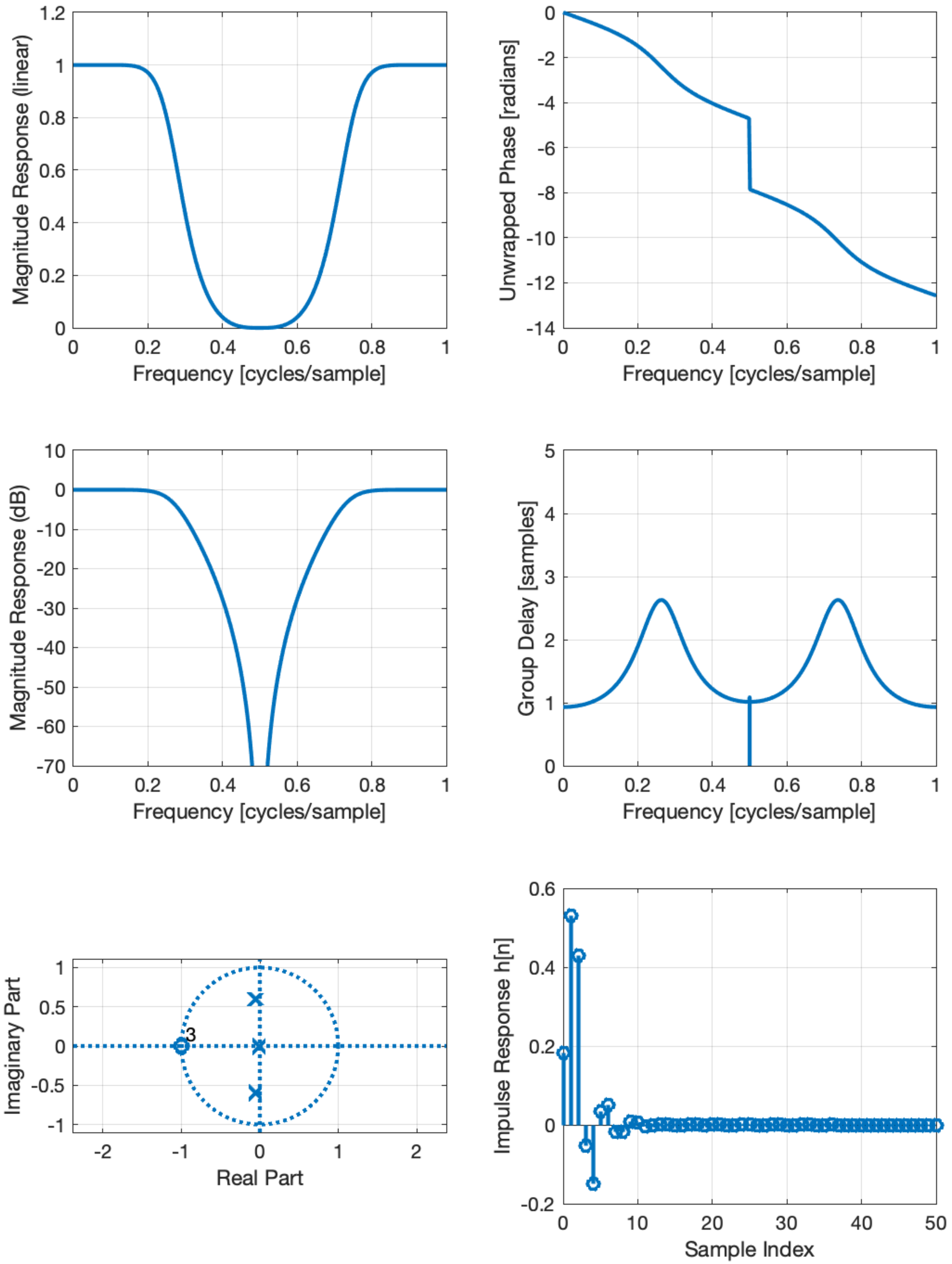
60 Hz Notch
Fs = 500 samples/second
Pole radius = 0.99
Pole angle = 2pi * 60/500



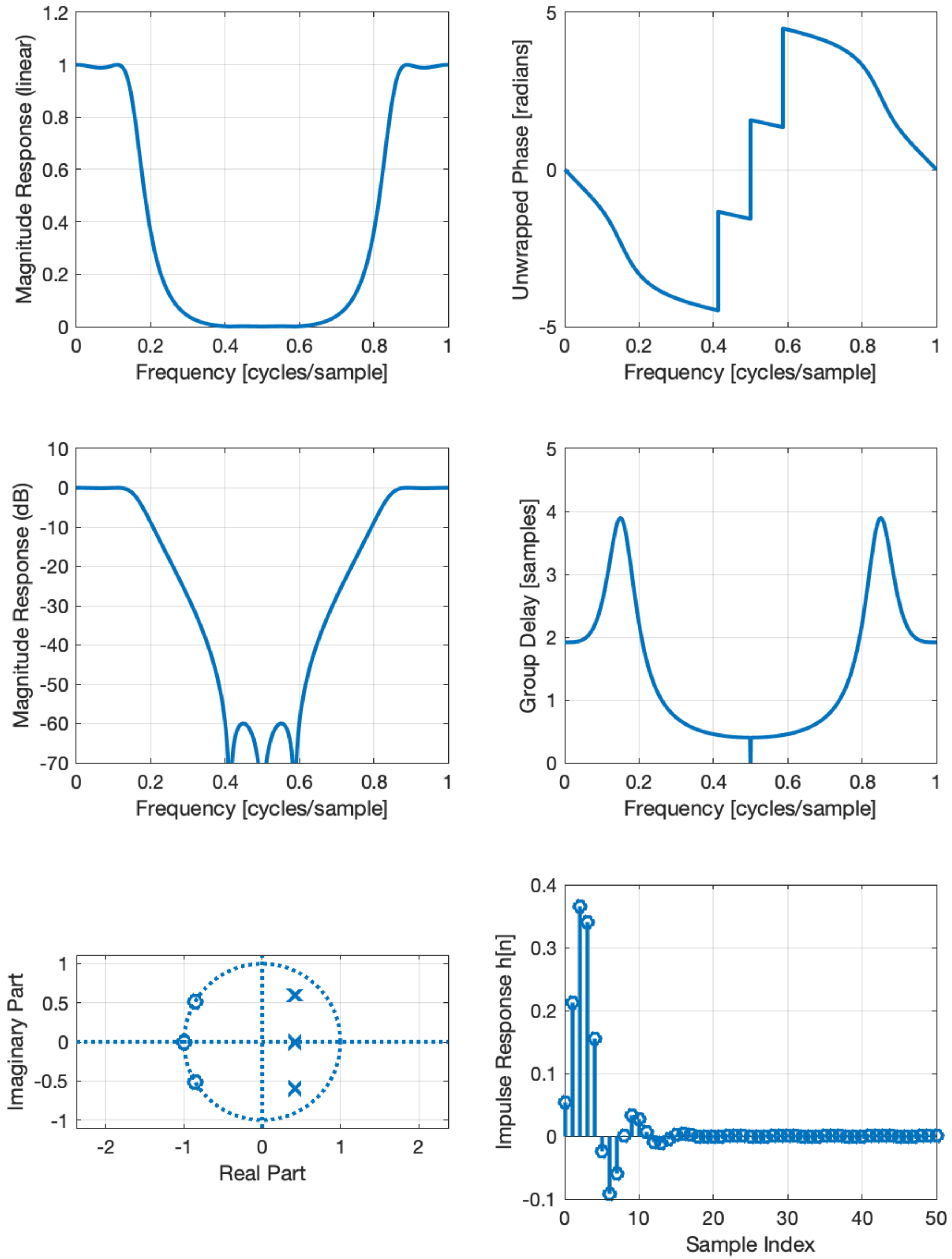
IIR Low Pass Filter
Butterworth Filter Design
3rd Order Filter



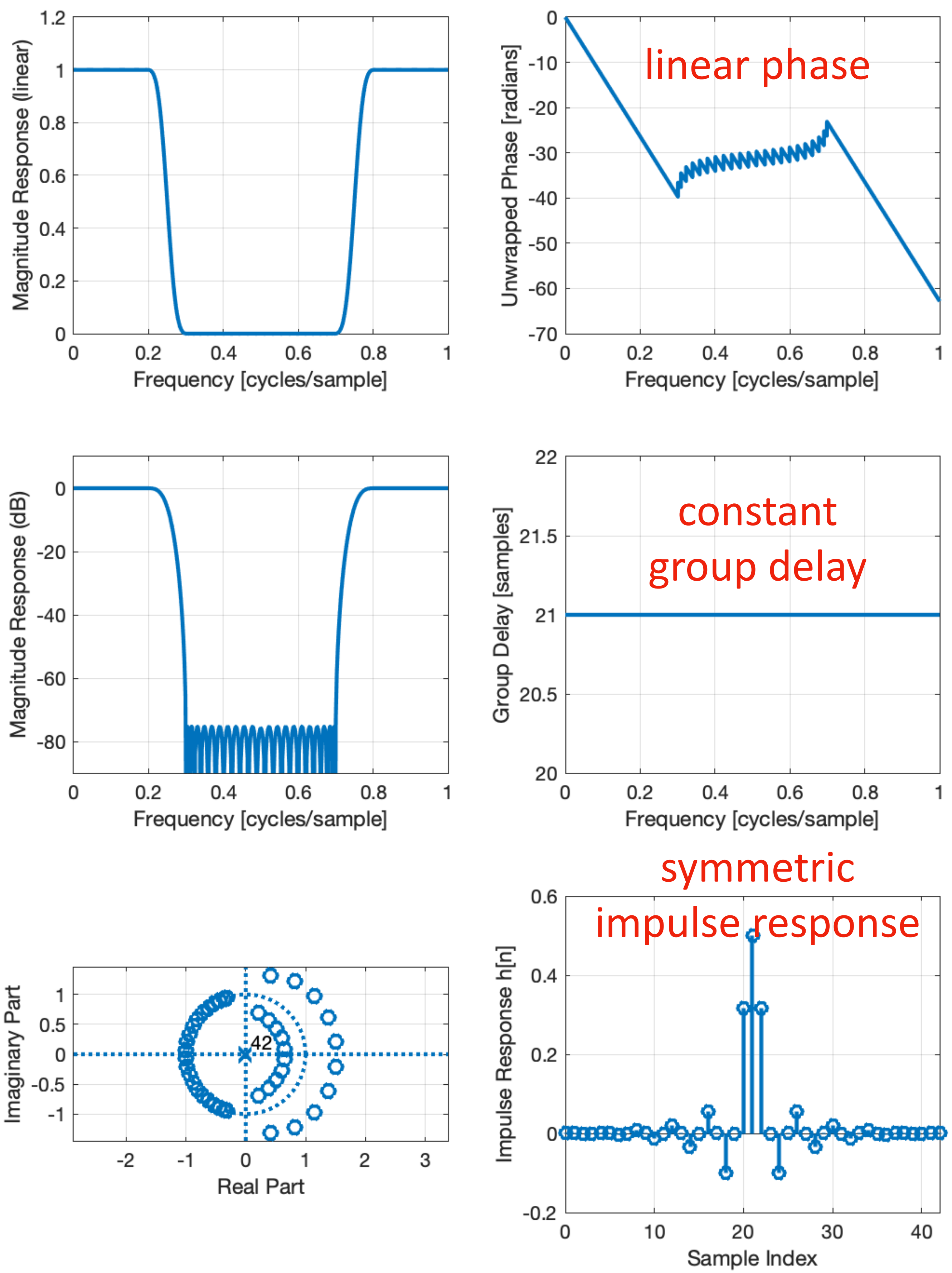
IIR Low Pass Filter
Chebyshev Filter Design
3rd Order Filter



IIR Low Pass Filter
Elliptic Filter Design
3rd Order Filter



FIR Low Pass Filter
Equiripple Filter Design
42 Order Filter



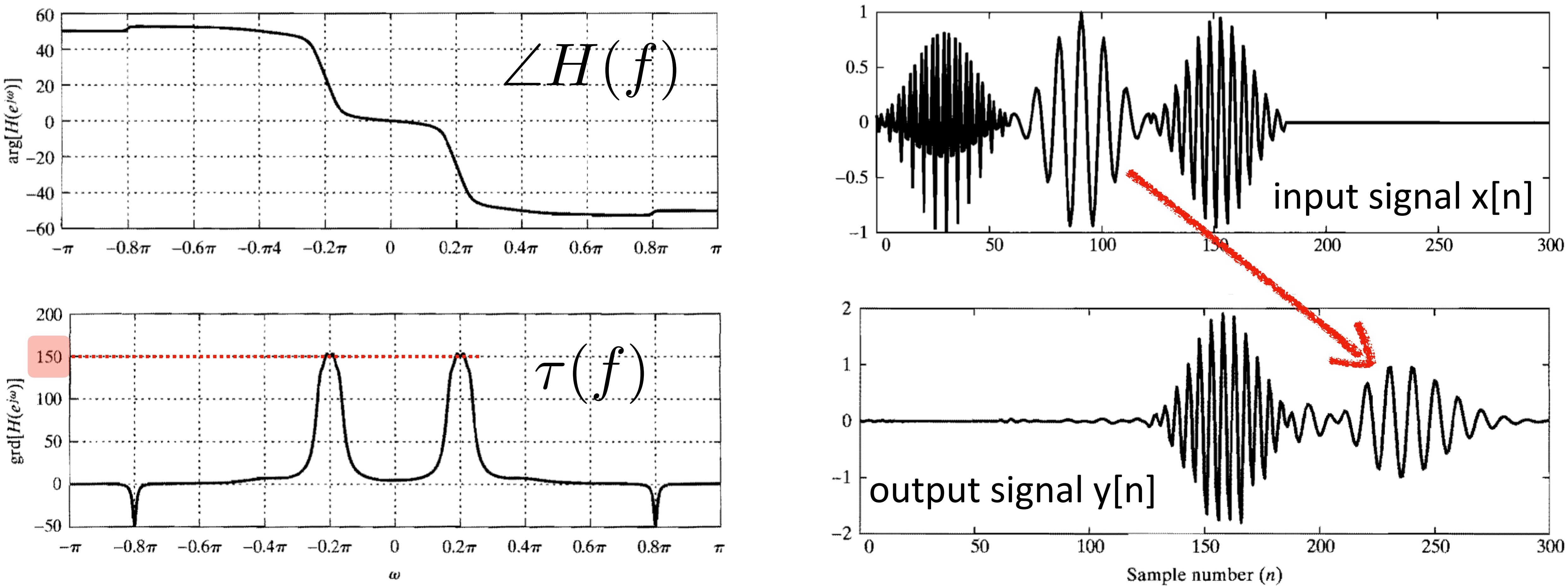
linear phase

constant group delay

symmetric impulse response

Generalized Linear Phase Systems

Remember this example of a non-linear phase system H(z)?



The three signal components each experience different delay.
This can’t happen with generalized linear phase systems which have constant group delay.

$$H(f) = A(f)e^{-j2\pi f\alpha + j\beta}$$

(A(f) is real bipolar function)

$$\angle H(f) = \text{sign}\{A(f)\} - 2\pi f\alpha + \beta$$

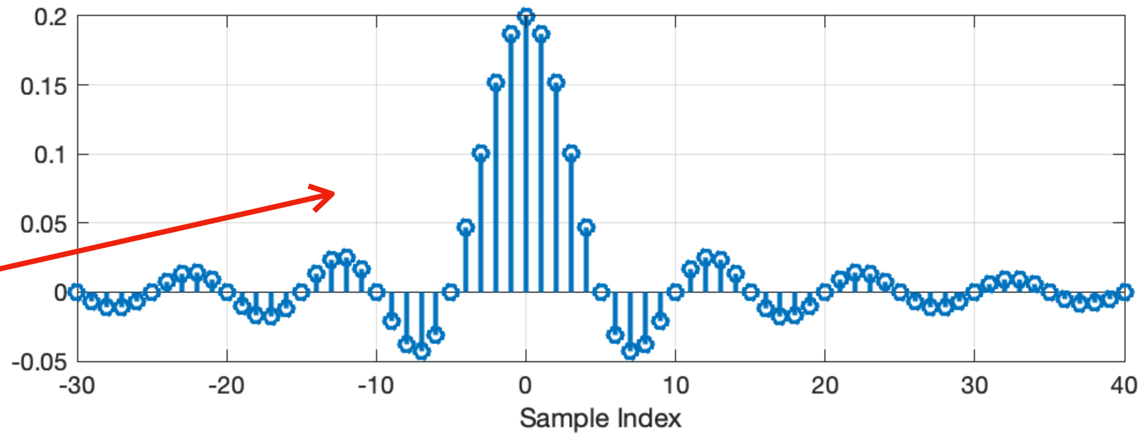
$$\tau(f) = \alpha \quad \text{(constant)}$$

Generalized Linear Phase Systems

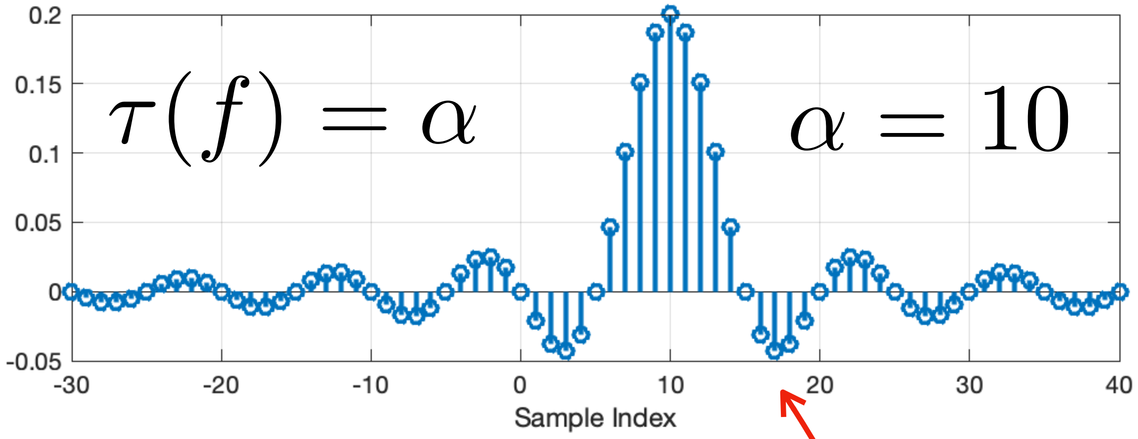
Ideal low pass filter with zero phase

$$H(f) = \begin{cases} 1, & |f| < f_c \\ 0, & f_c < |f| < \frac{1}{2} \end{cases}$$

$$h[n] = \frac{\sin(2\pi f_c n)}{\pi n}$$



$$f_c = 0.1$$



Ideal low pass filter with linear phase

$$H(f) = \begin{cases} e^{-j2\pi f\alpha}, & |f| < f_c \\ 0, & f_c < |f| < \frac{1}{2} \end{cases} = e^{-j2\pi f\alpha} \begin{cases} 1, & |f| < f_c \\ 0, & f_c < |f| < \frac{1}{2} \end{cases}$$

$$h[n] = \frac{\sin(2\pi f_c[n-\alpha])}{\pi[n-\alpha]}$$

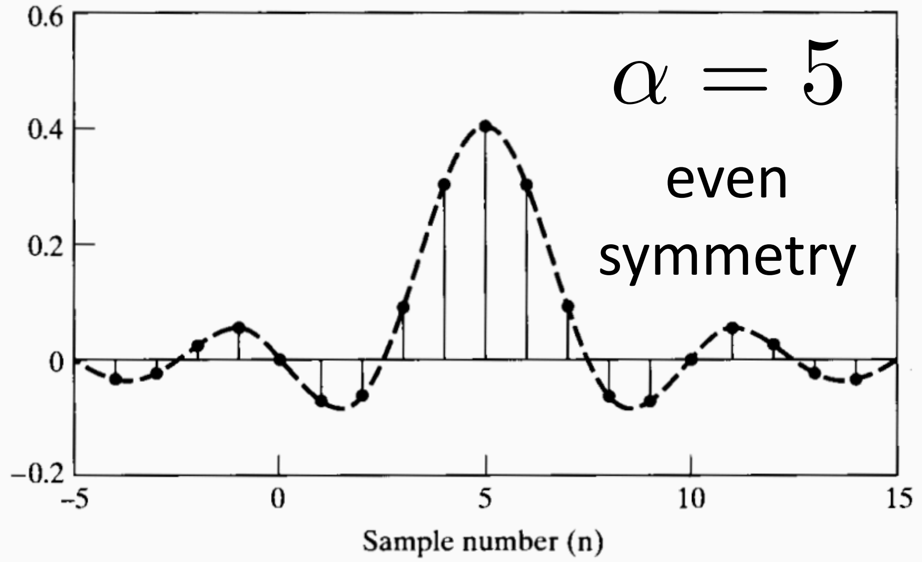
$$\tau(f) = \alpha$$

Generalized Linear Phase Systems

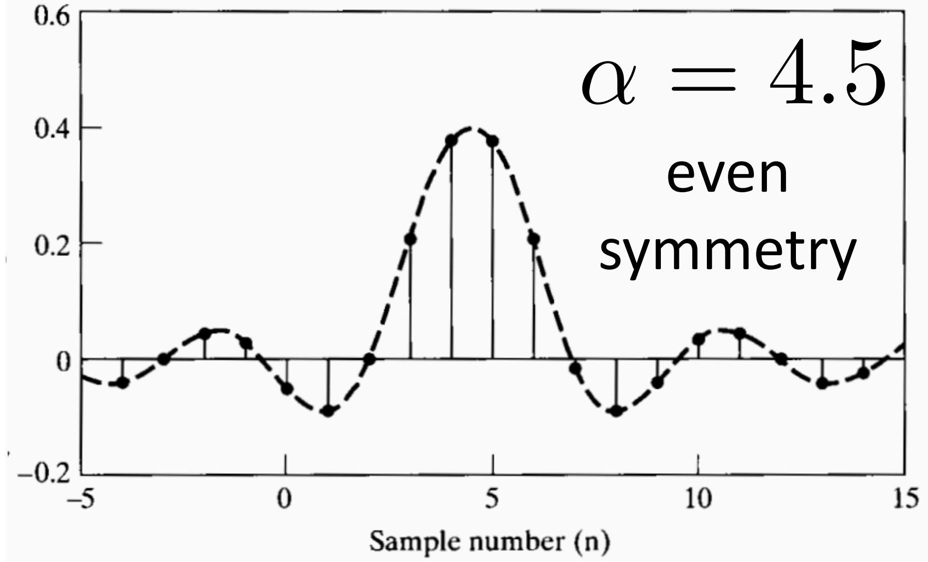
$$H(f) = \begin{cases} e^{-j2\pi f \alpha}, & |f| < f_c \\ 0, & f_c < |f| < \frac{1}{2} \end{cases}$$

$$= e^{-j2\pi f \alpha} \begin{cases} 1, & |f| < f_c \\ 0, & f_c < |f| < \frac{1}{2} \end{cases}$$

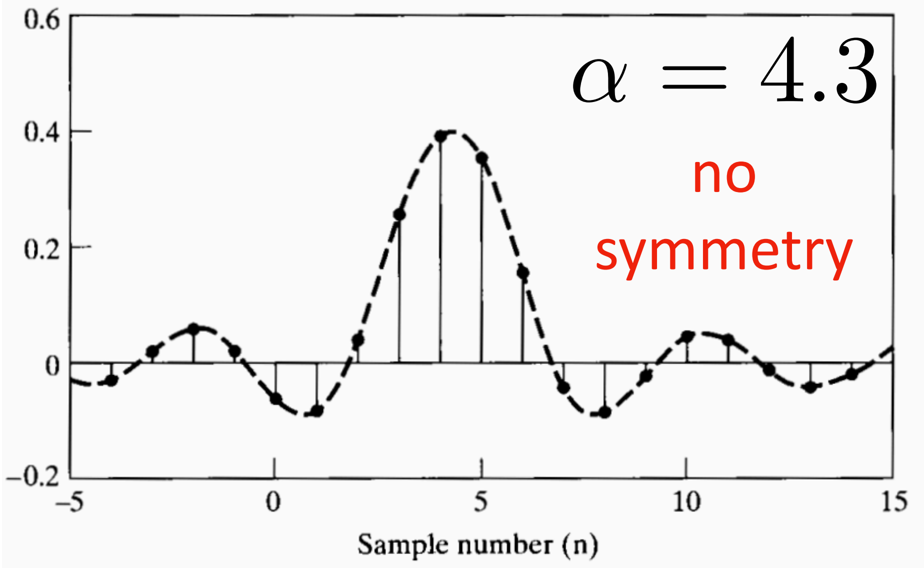
$$h[n] = \frac{\sin(2\pi f_c [n - \alpha])}{\pi [n - \alpha]}$$



(a)



(b)



Generalized Linear Phase Systems: Four Interesting Cases

Impulse response:
h[n] , n = 0, 1, 2, ..., M,
length = M+1

$$H(f) = A(f)e^{-j2\pi f\alpha + j\beta}$$

$$\angle H(f) = \text{sign}\{A(f)\} - 2\pi f\alpha + \beta$$

$$\tau(f) = \alpha \quad \text{(constant)}$$

Type	Symmetry	Order M	b, A(f)
1	h[n] = h[M-n] even about a=M/2	M = even a = int	b = 0 or pi A(f) real & even
2	h[n] = h[M-n] even about a=M/2	M = odd a = int + 0.5	b = 0 or pi A(f) real & even
3	h[n] = -h[M-n] odd about a=M/2	M = even a = int	b = pi/2 or 3pi/2 A(f) real & odd
4	h[n] = -h[M-n] odd about a=M/2	M = odd a = int + 0.5	b = pi/2 or 3pi/2 A(f) real & odd

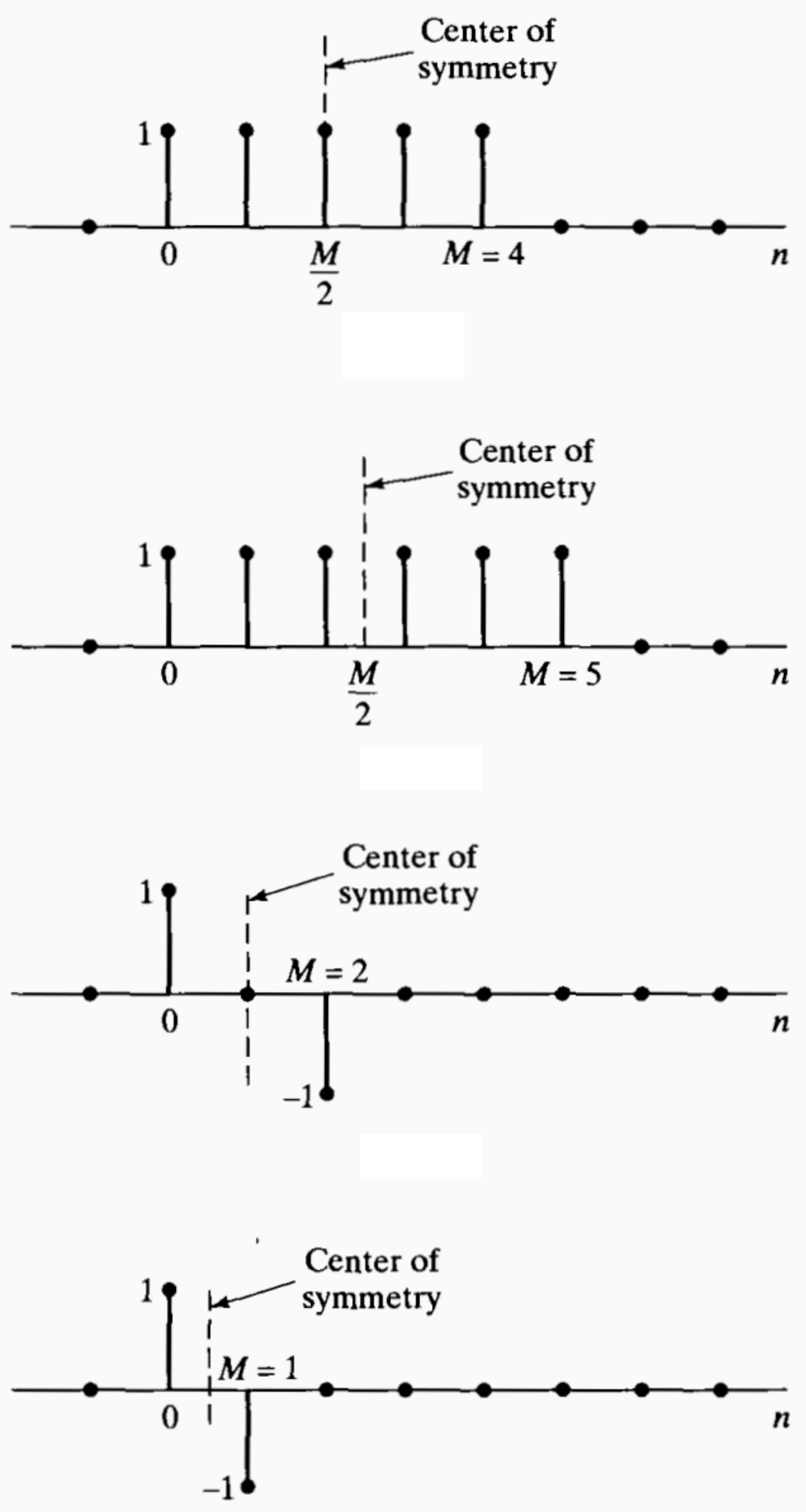
What is the z-transform?

Type 1: M=even, h[n]=even

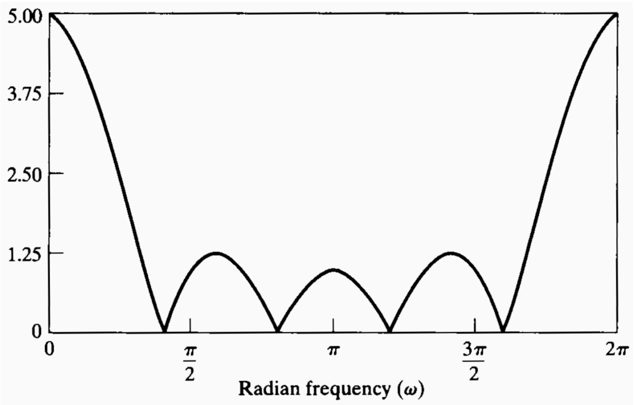
Type 2: M=odd, h[n]=even

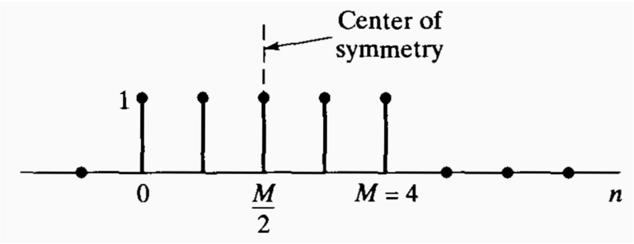
Type 3: M=even, h[n]=odd

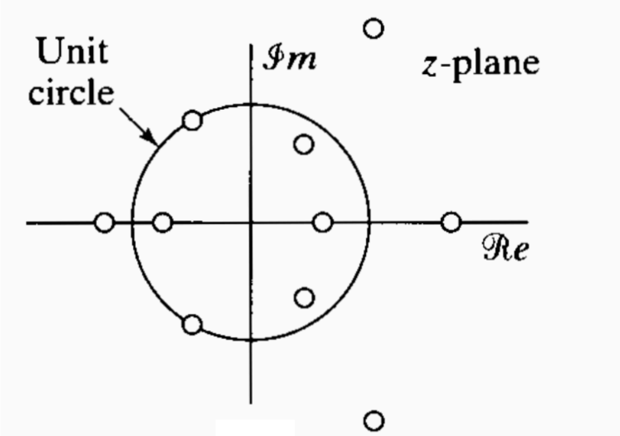
Type 4: M=odd, h[n]=odd



All these systems have linear phase and constant group delay

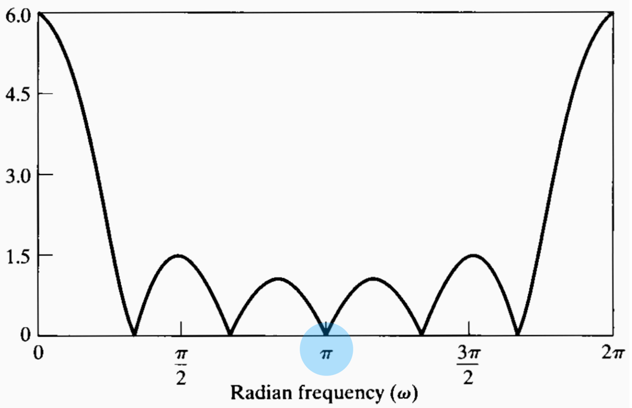


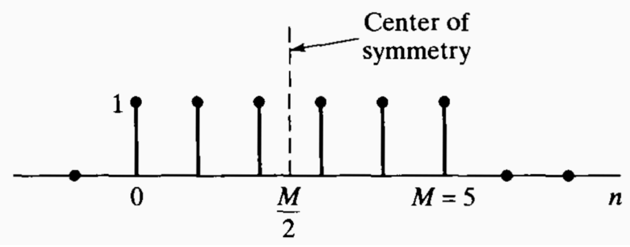


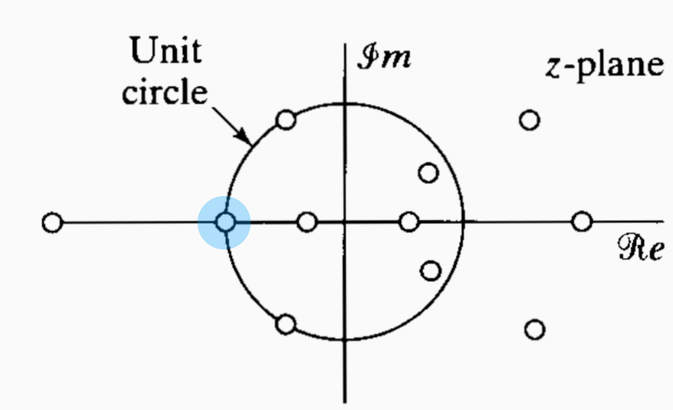


Type 1:
M=even,
h[n]=even

(any kind of filter)



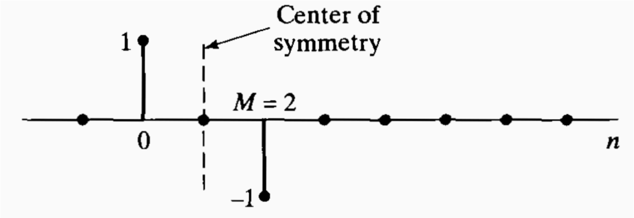


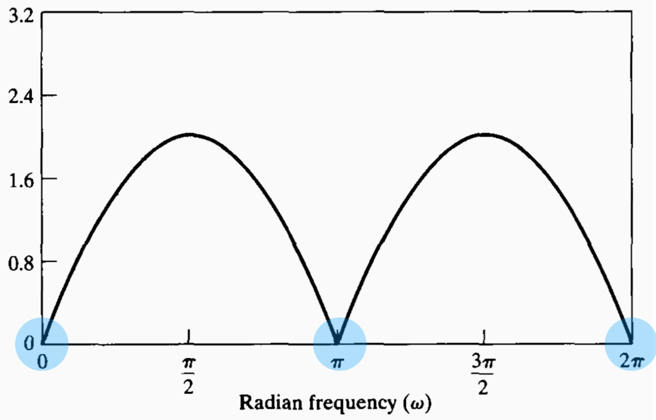


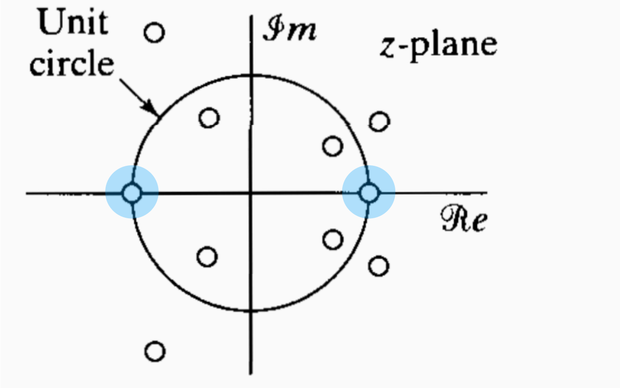
Type 2:
M=odd,
h[n]=even

(low pass filter)

(bandpass filter, Hilbert transformer)

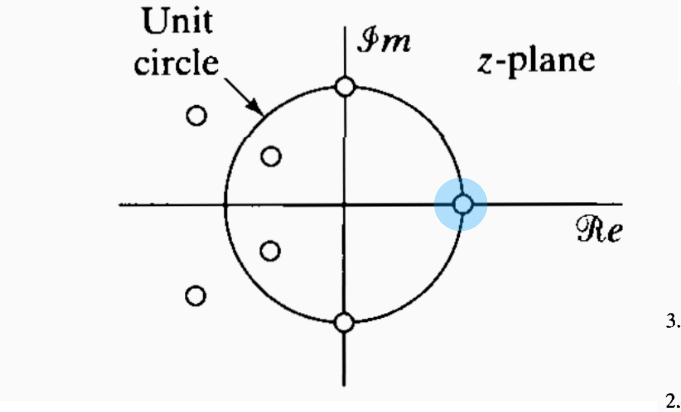


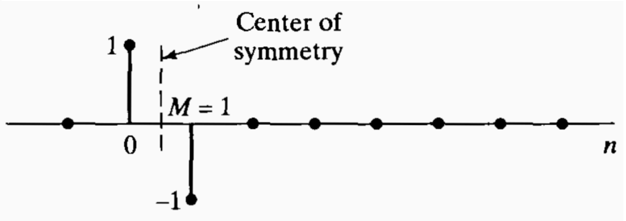


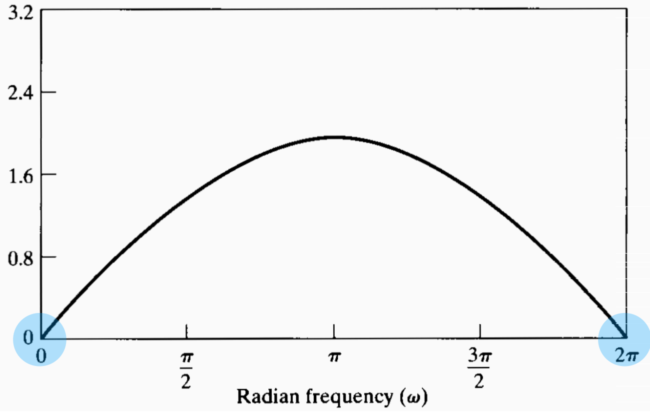


Type 3:
M=even,
h[n]=odd

(high pass filter, differentiator)







Type 4:
M=odd,
h[n]=odd

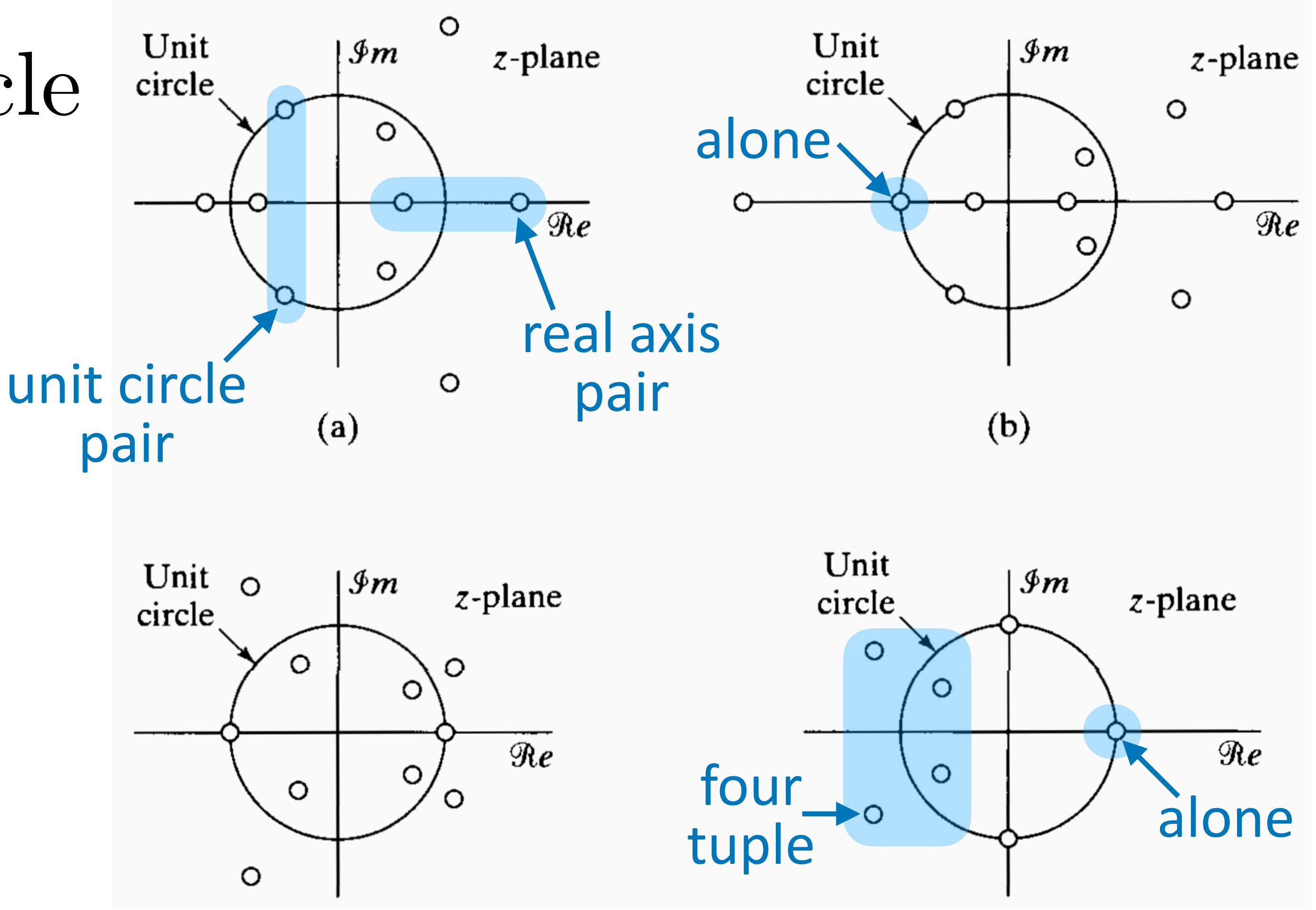
Roots of Real Polynomials

Generalized linear phase systems are: (1) FIR and (2) symmetric (even or odd).

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[M]z^{-M}$$

Zero locations have special structure due to real coefficients $h[n]$ and symmetry.

$$\begin{array}{rcl} z, z^*, z^{-1}, z^{-*} & \leftrightarrow & \text{general location} \\ z, z^* & \leftrightarrow & \text{unit circle} \\ z, z^{-1} & \leftrightarrow & \text{real line} \\ \pm 1 & \leftrightarrow & \text{alone} \end{array}$$



All poles are at the origin.

Notice that none of these FIR systems is minimum phase.

The End

Now let’s go design some filters!