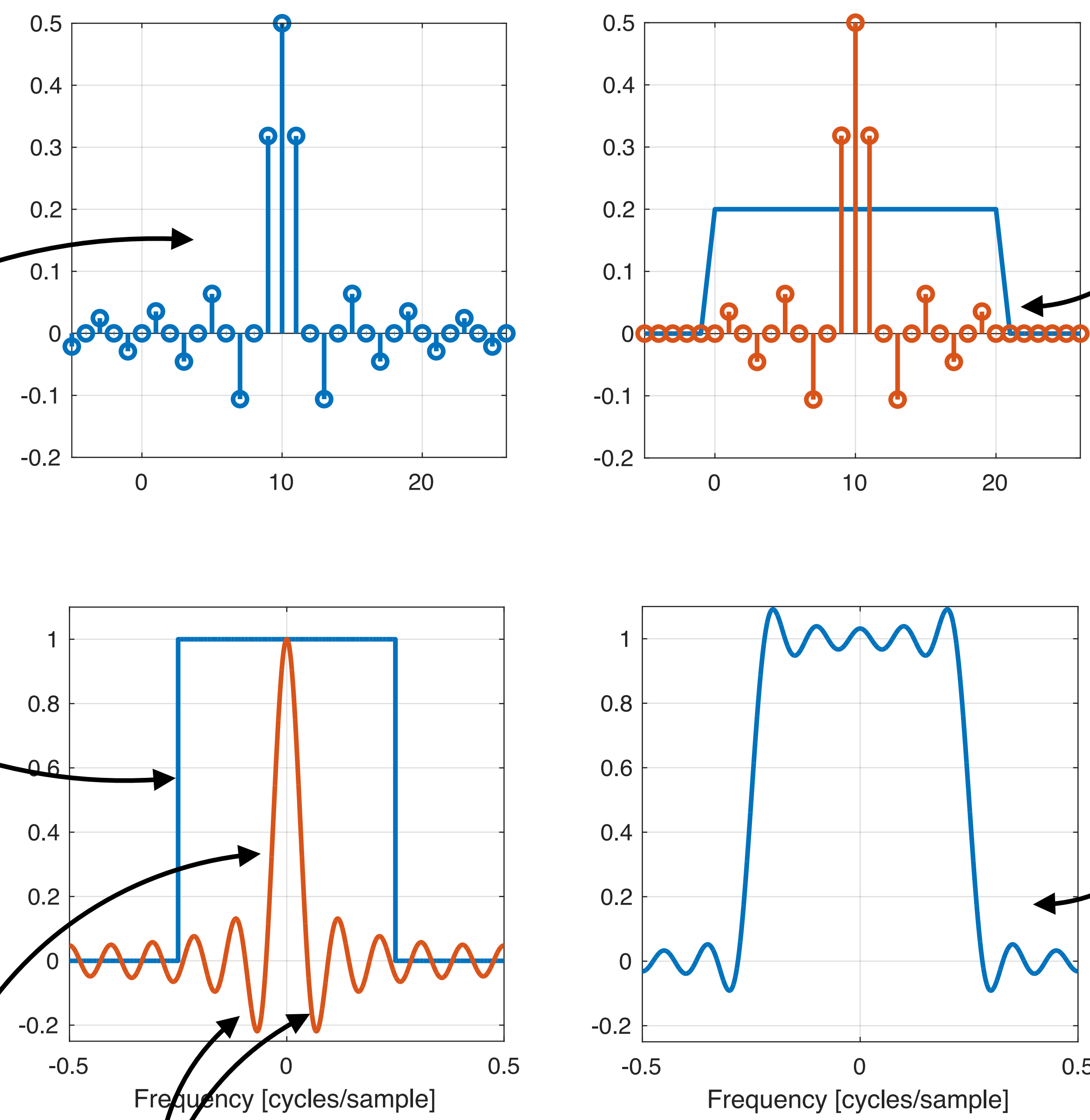
FIR Filter Design:
The Windowing Method

ECE 3640 Discrete-Time Signals and Systems
Utah State University
Spring 2020

Periodic Convolution in Frequency Domain

$$h[n] = h_i[n] \cdot w[n] \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad H(f) = H_i(f) \circledast W(f)$$



multiply by window in the time domain

(w[n] has length and shape)

response of ideal filter

periodic convolution with window in the frequency domain

W(f) has main lobe and side lobes

H(f) has ripples in pass and stop bands and a transition band

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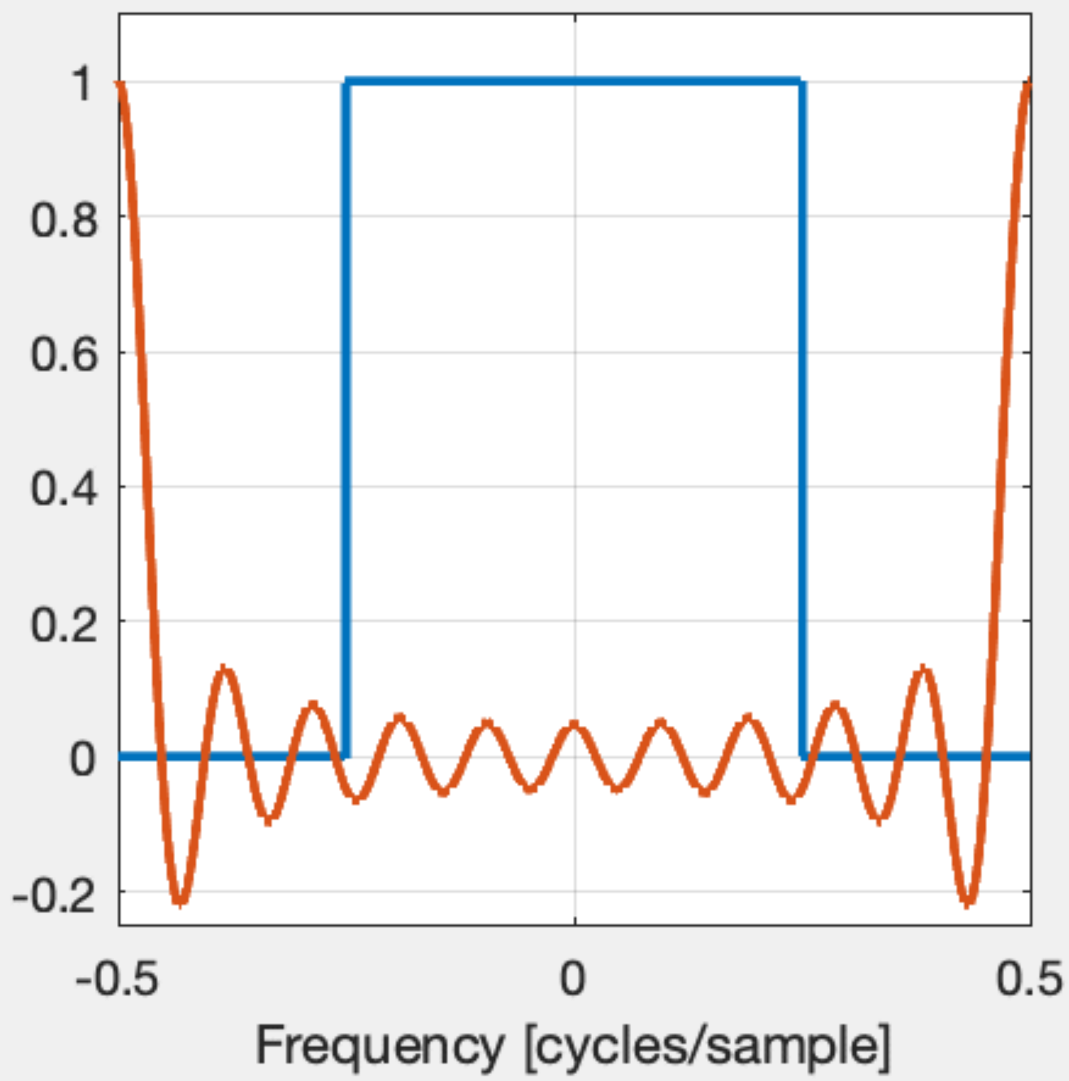
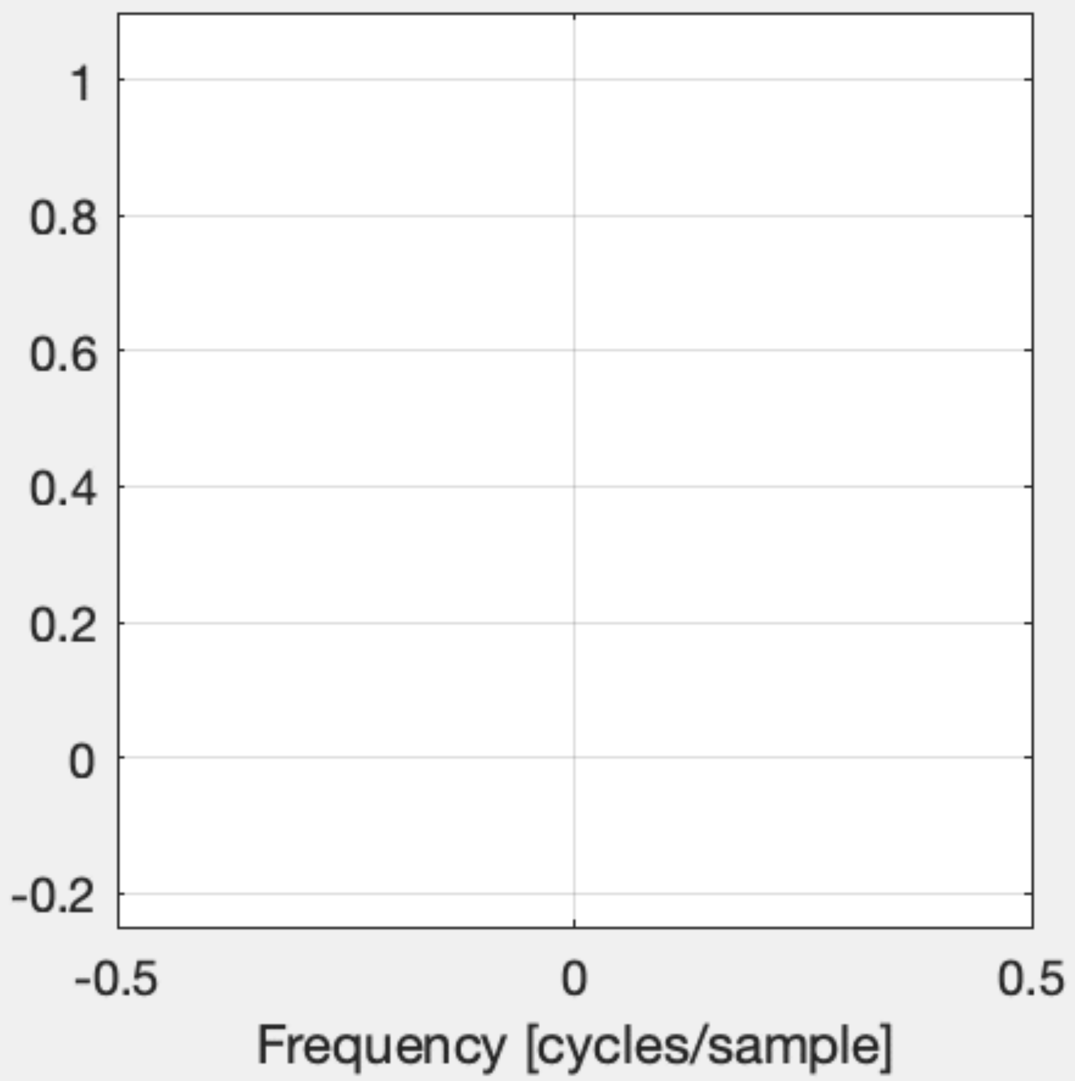
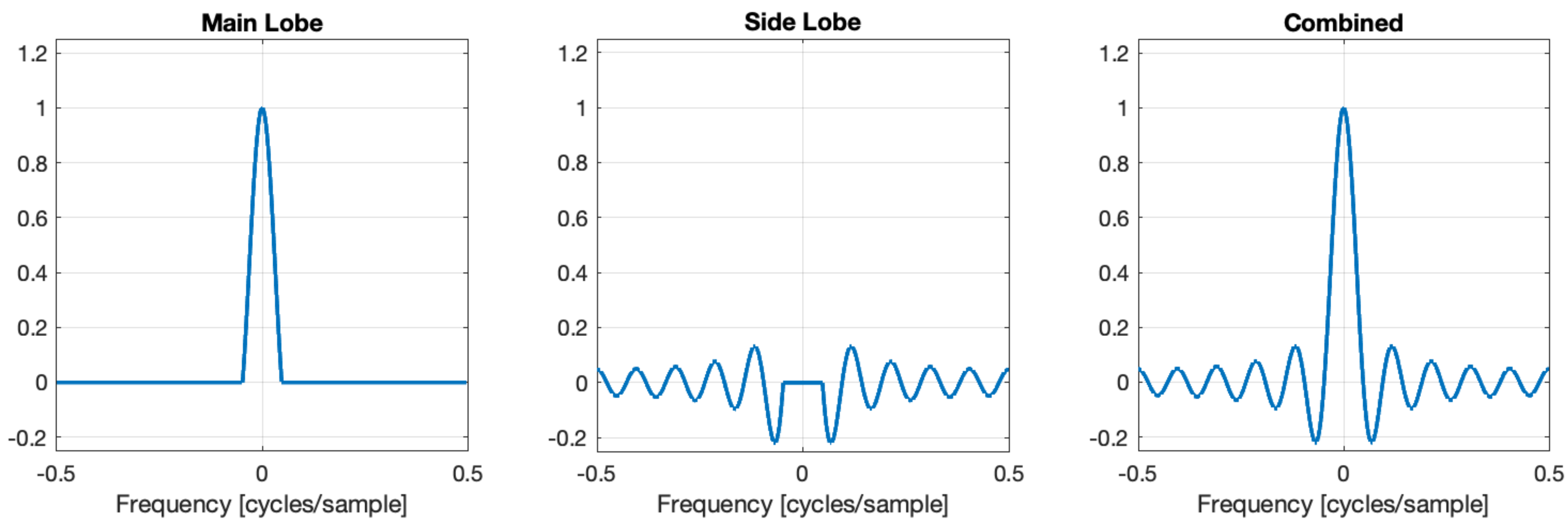




Periodic Convolution in Frequency Domain

$$h[n] = h_i[n] \cdot w[n] \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad H(f) = H_i(f) \circledast W(f)$$

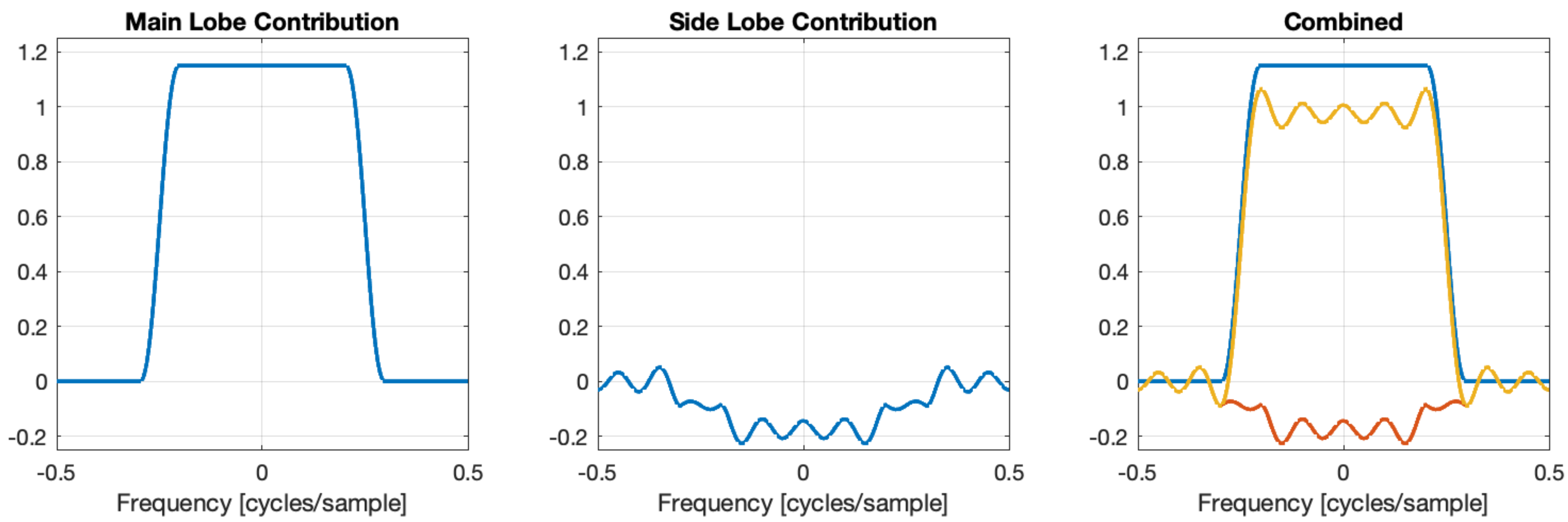
$$W(f) = W_{\text{main}}(f) + W_{\text{side}}(f)$$



Periodic Convolution in Frequency Domain

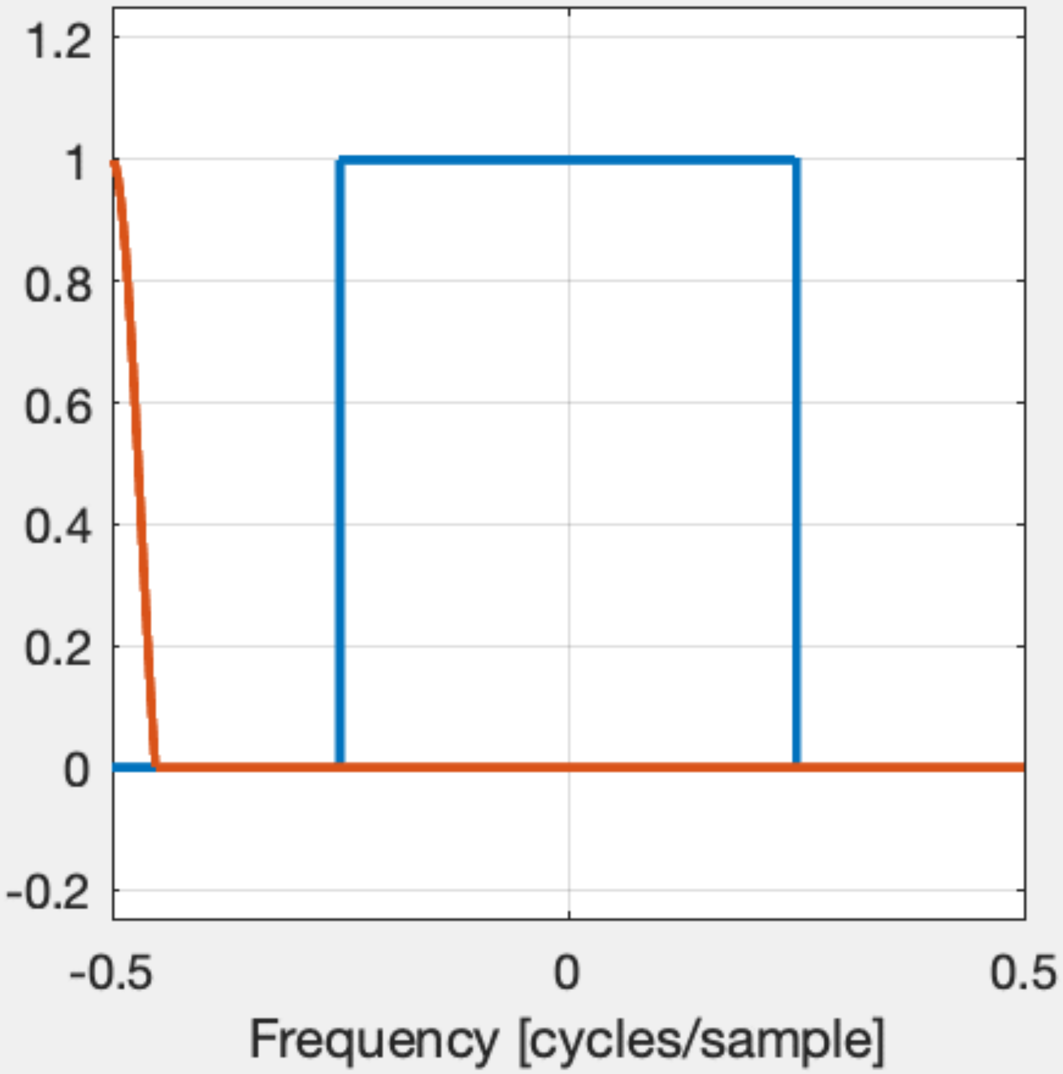
$$h[n] = h_i[n] \cdot w[n] \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad H(f) = H_i(f) \circledast W(f)$$

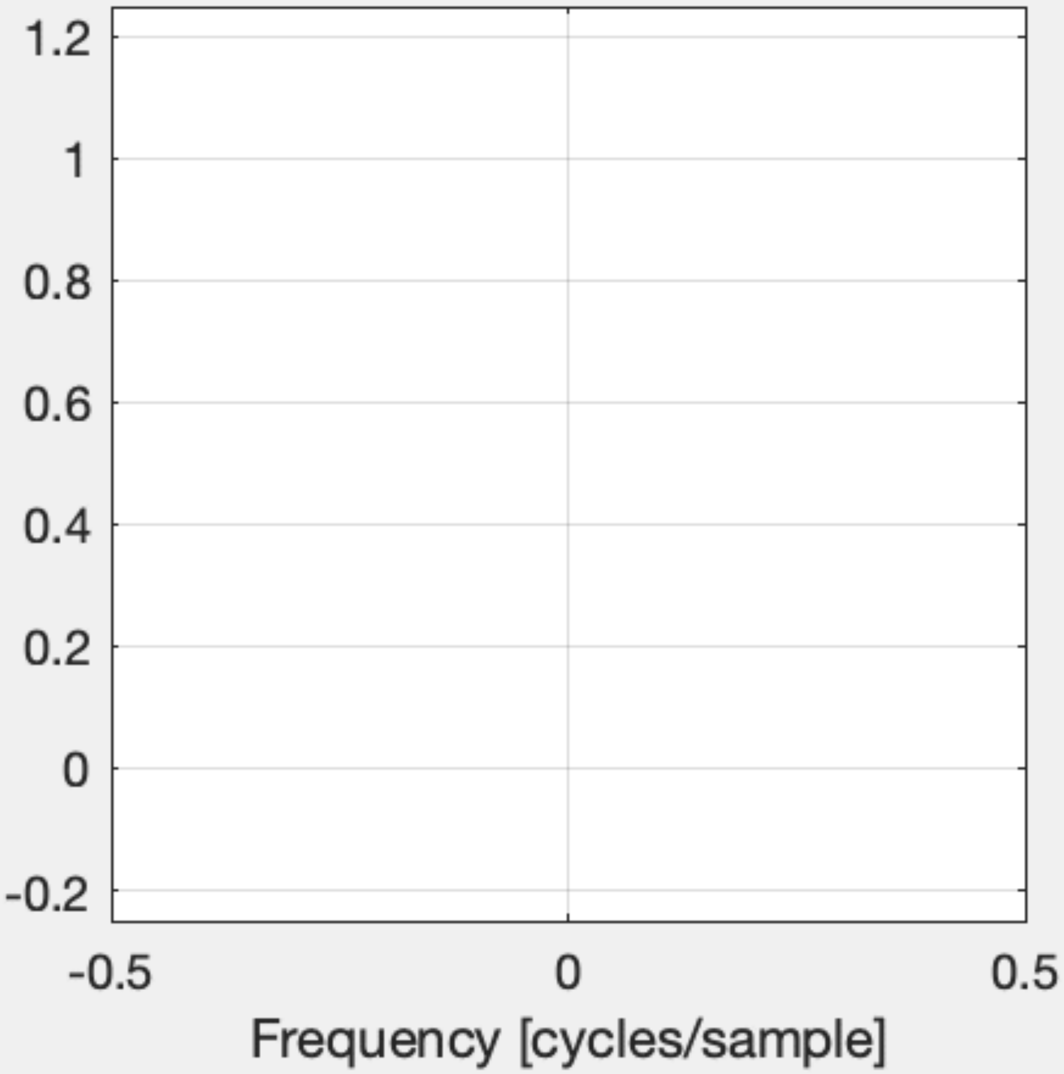
$$\begin{aligned} H(f) &= H_i(f) \circledast W(f) \\ &= H_i(f) \circledast (W_{\text{main}}(f) + W_{\text{side}}(f)) \\ &= H_i(f) \circledast W_{\text{main}}(f) + H_i(f) \circledast W_{\text{side}}(f) \\ &= H_{i,\text{main}}(f) + H_{i,\text{side}}(f) \end{aligned}$$



Periodic Convolution in Frequency Domain

$$h[n] = h_i[n] \cdot w[n] \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad H(f) = H_i(f) \circledast W(f)$$





Periodic Convolution in Frequency Domain

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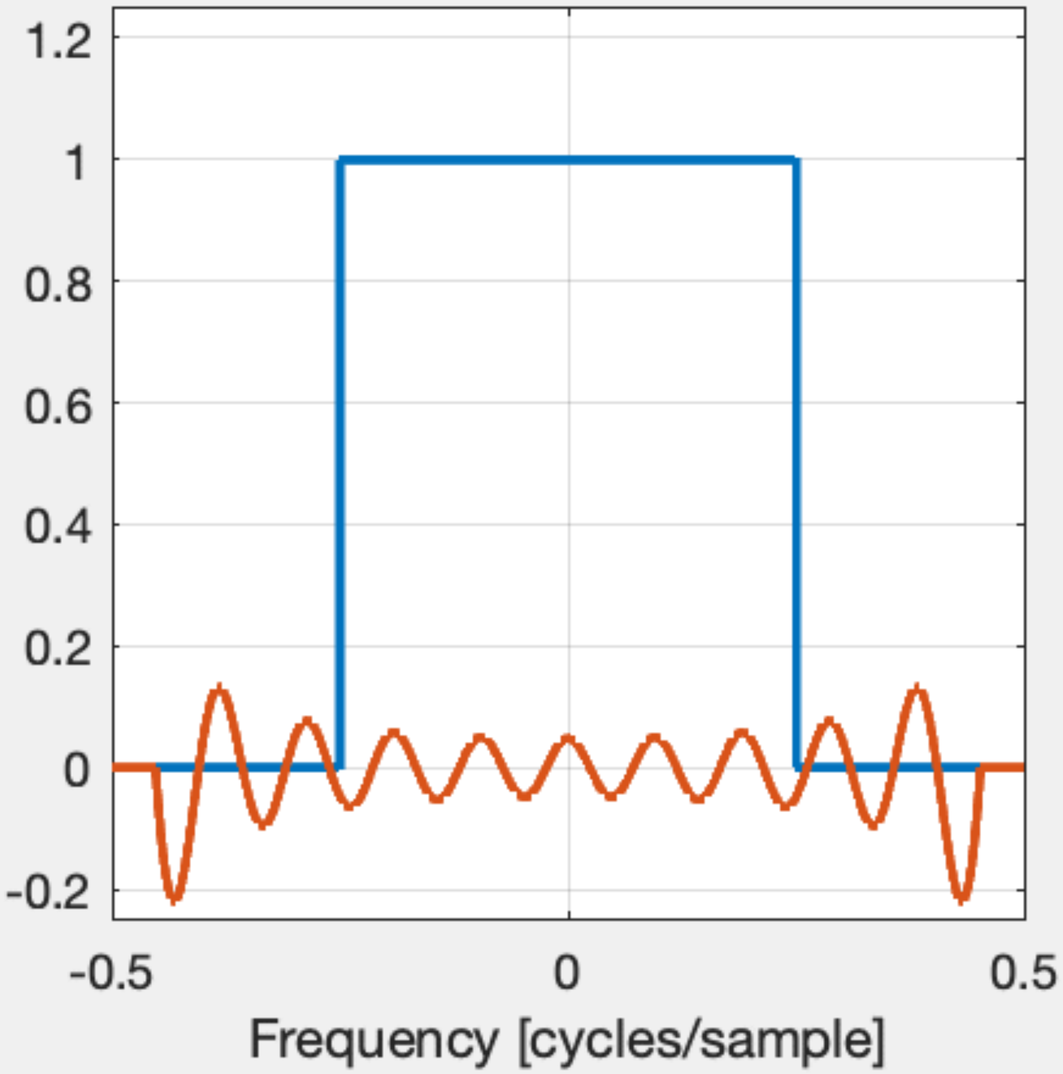
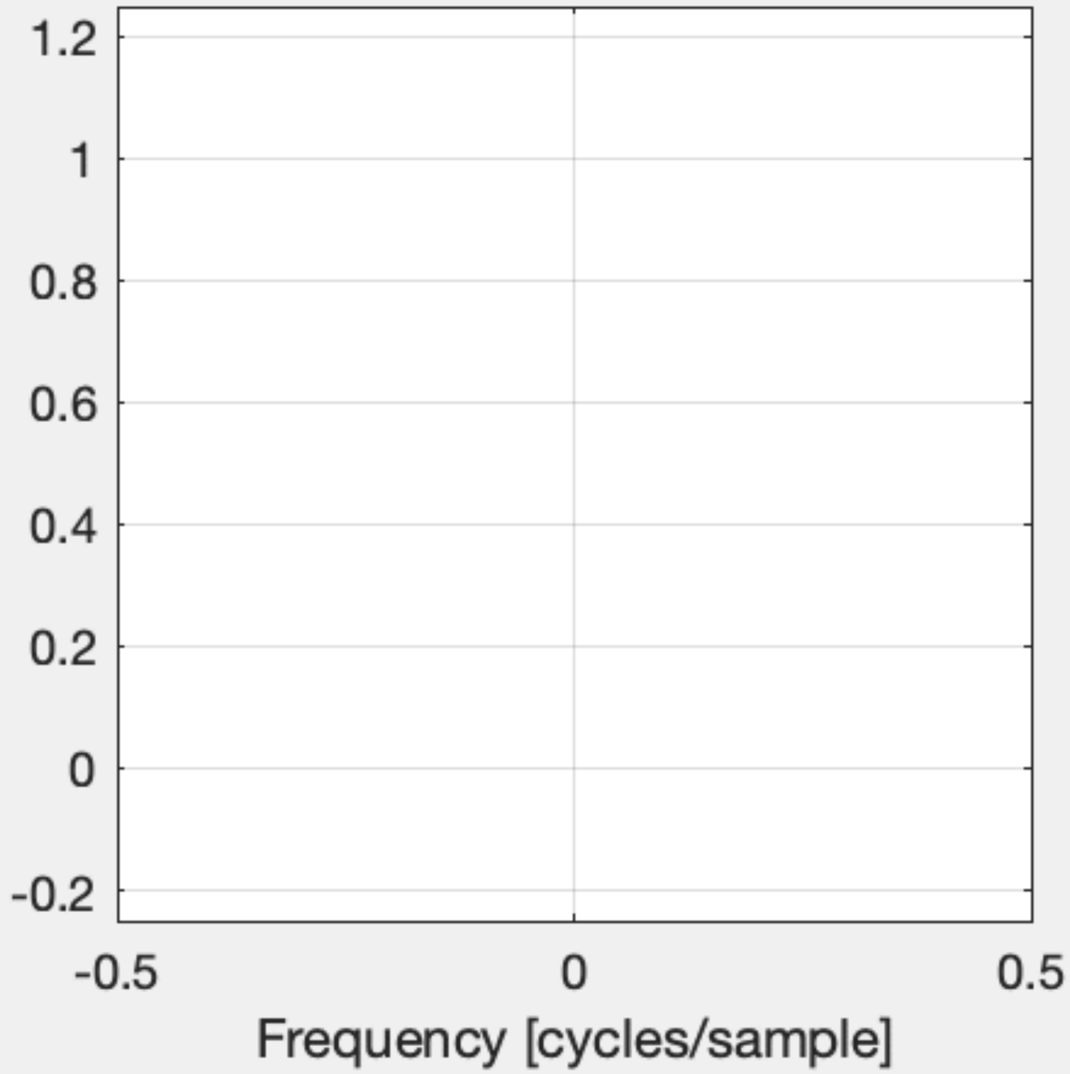
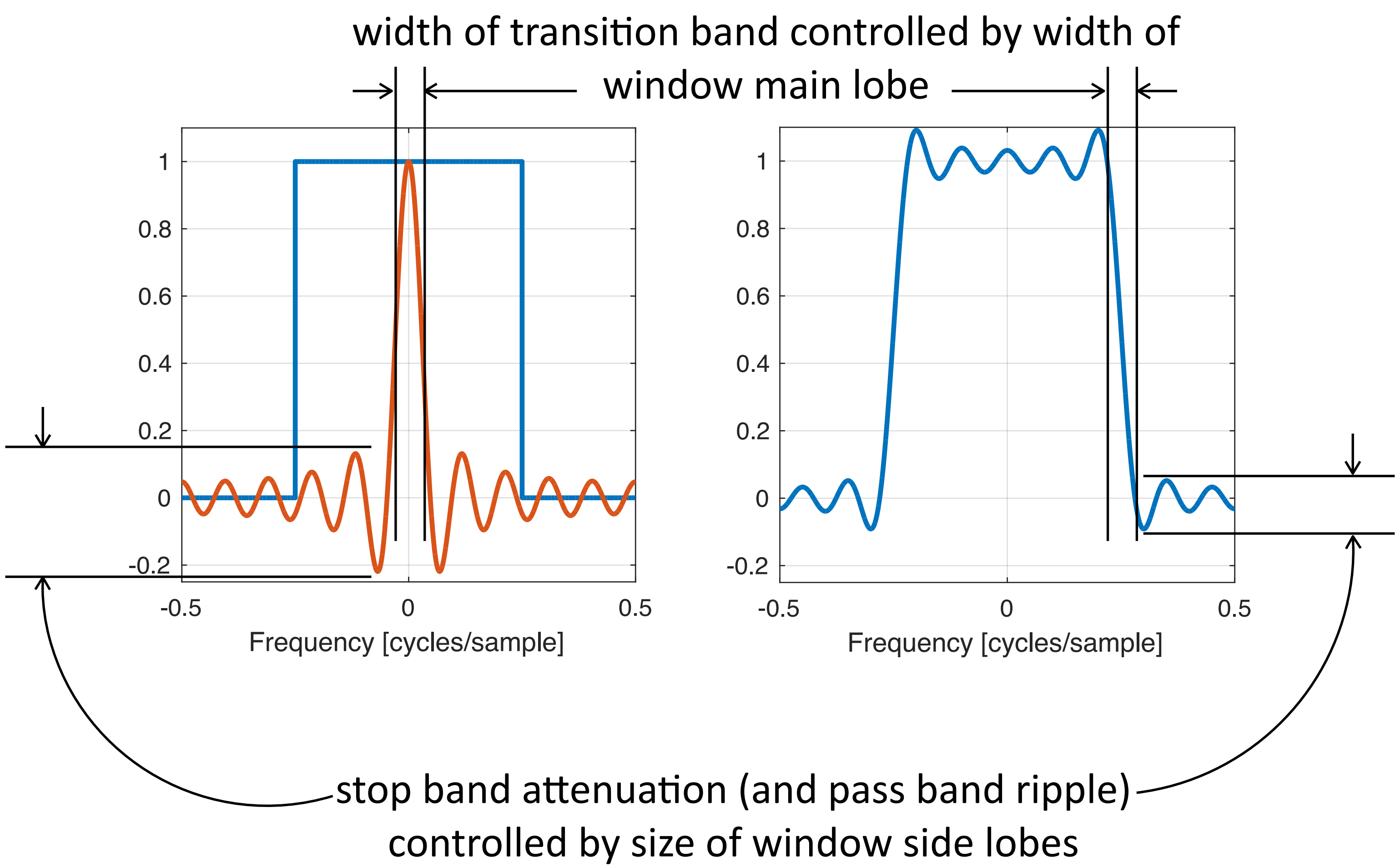




Important Properties of Window Functions

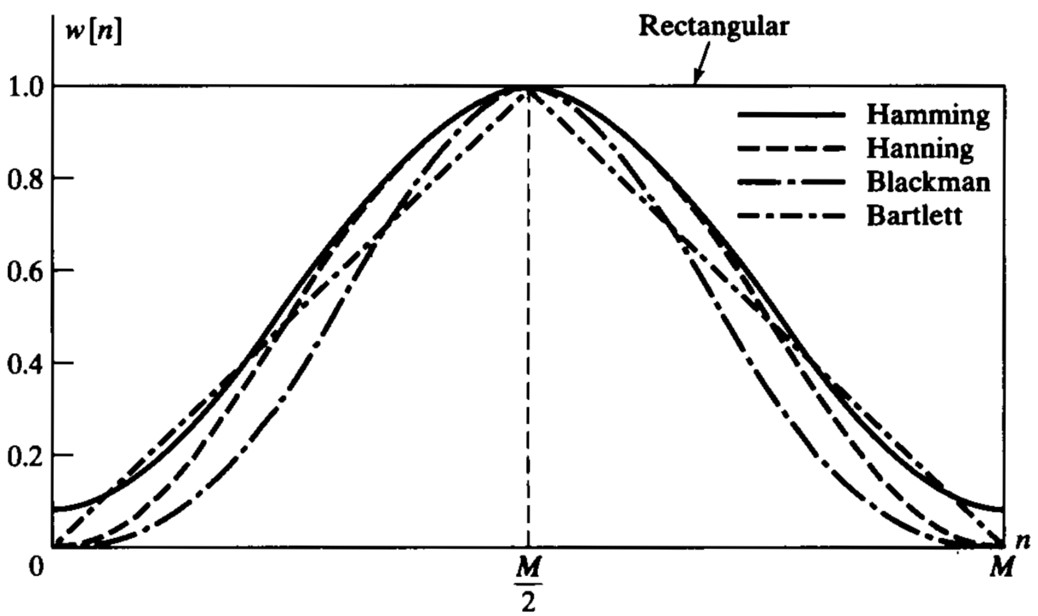
$$h[n] = h_i[n] \cdot w[n] \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad H(f) = H_i(f) \circledast W(f)$$



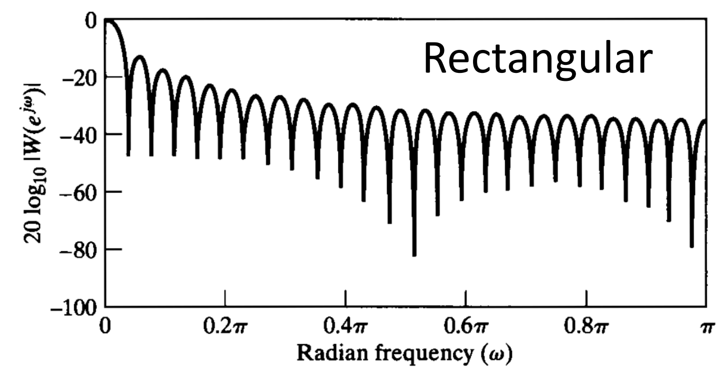
Window main lobe width controlled by shape and length of window w[n].

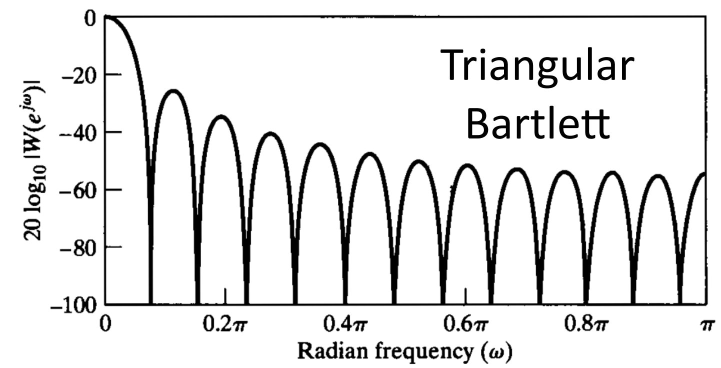
Window side lobe height controlled by shape of window w[n].

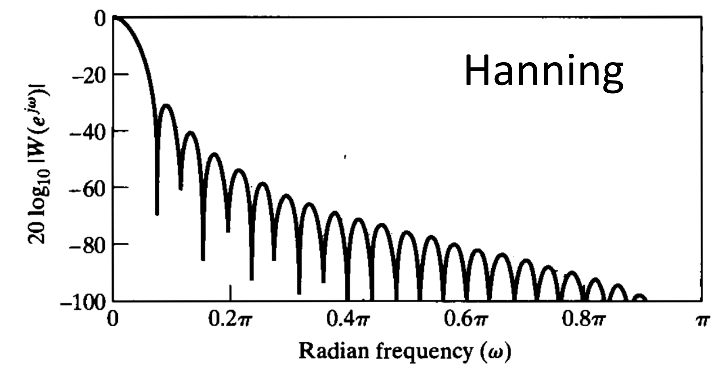
Examples of Window Functions

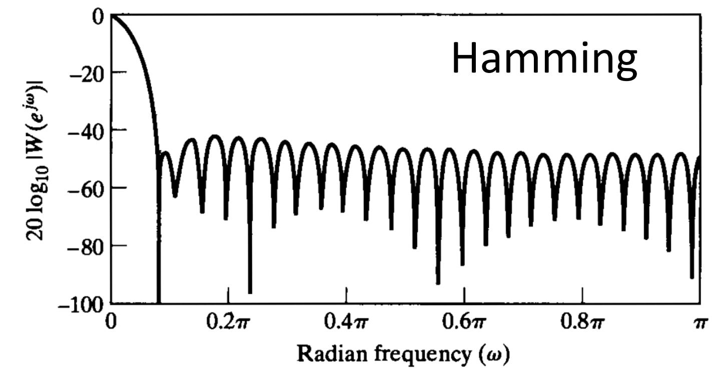


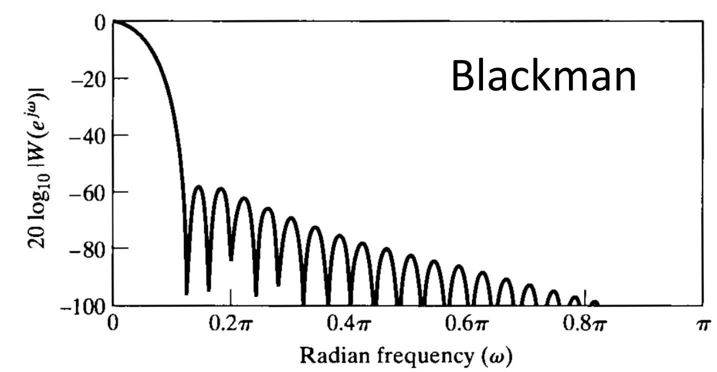
commonly used window shapes











Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Window main lobe width controlled by shape and length of window w[n].

Window side lobe height controlled by shape of window w[n].

Generalized Linear Phase Systems: Four Interesting Cases

Impulse response:
h[n] , n = 0, 1, 2, ..., M,
length = M+1

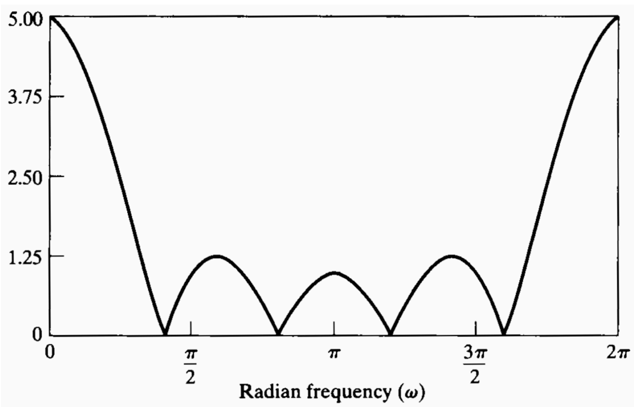
$$H(f) = A(f)e^{-j2\pi f\alpha + j\beta}$$

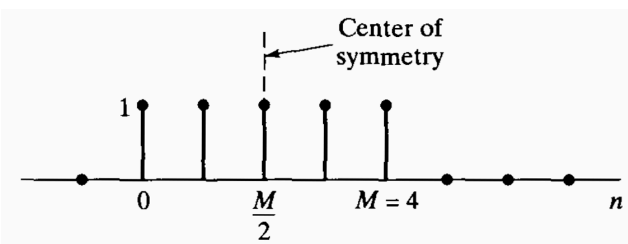
$$\angle H(f) = \text{sign}\{A(f)\} - 2\pi f\alpha + \beta$$

$$\tau(f) = \alpha \quad \text{(constant)}$$

Type	Symmetry h[n]=h[M-n] (even) h[n]=-h[M-n] (odd)	Order M	b, A(f)
1	even	even (a = M/2 = int)	b = 0 or pi A(f) real & even
2	even	odd (a = M/2 = int + 0.5)	b = 0 or pi A(f) real & even
3	odd	even (a = M/2 = int)	b = pi/2 or 3pi/2 A(f) real & odd
4	odd	odd (a = M/2 = int + 0.5)	b = pi/2 or 3pi/2 A(f) real & odd

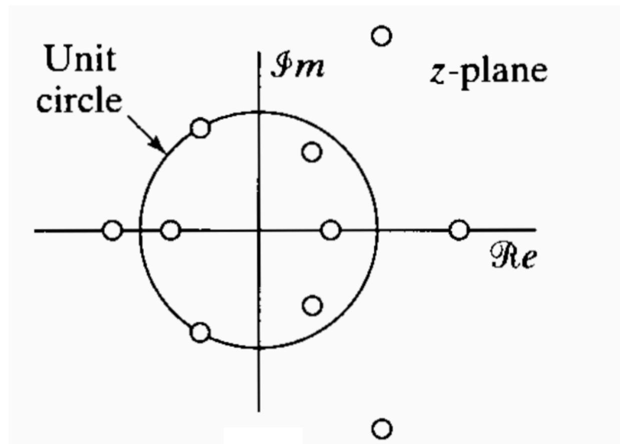
All these systems have linear phase and constant group delay



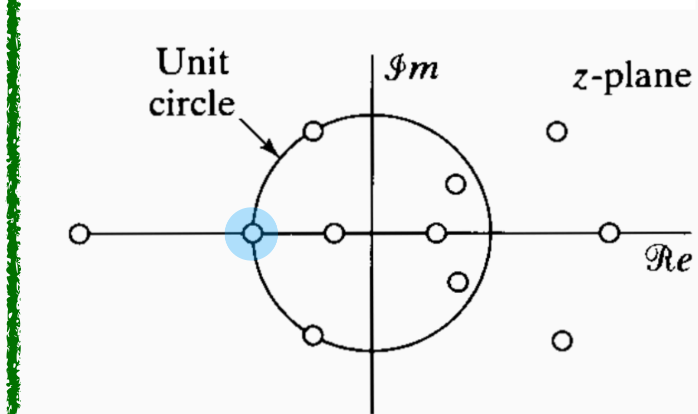


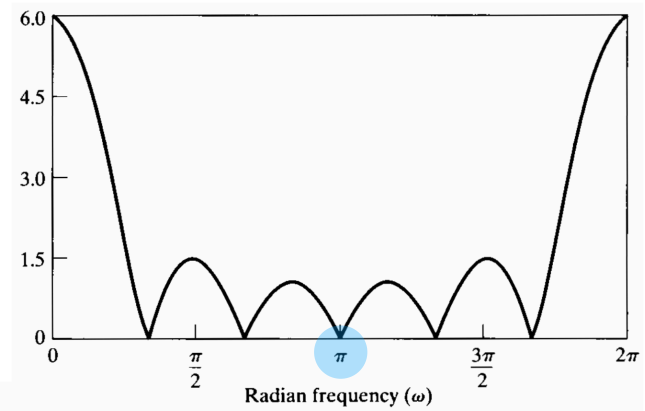
(any kind of filter)

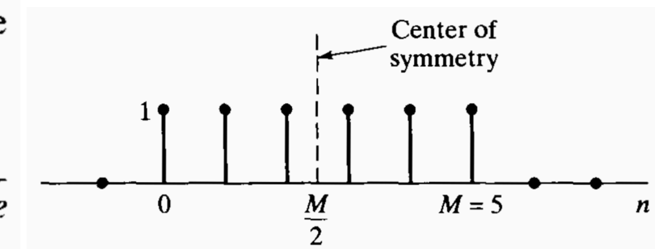
Type 1:
M=even,
h[n]=even



Type 2:
M=odd,
h[n]=even

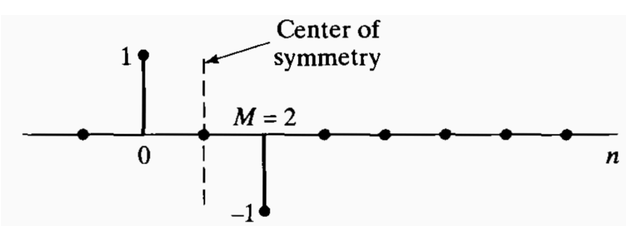


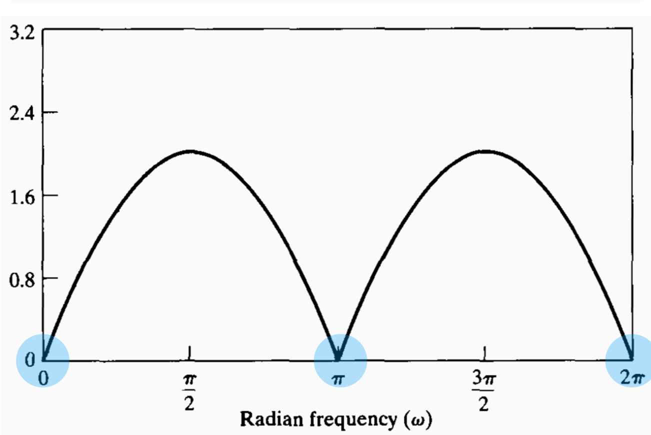


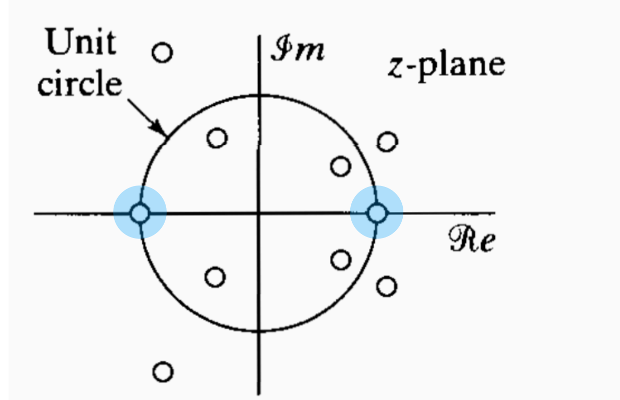


(low pass filter)

(bandpass filter, Hilbert transformer)

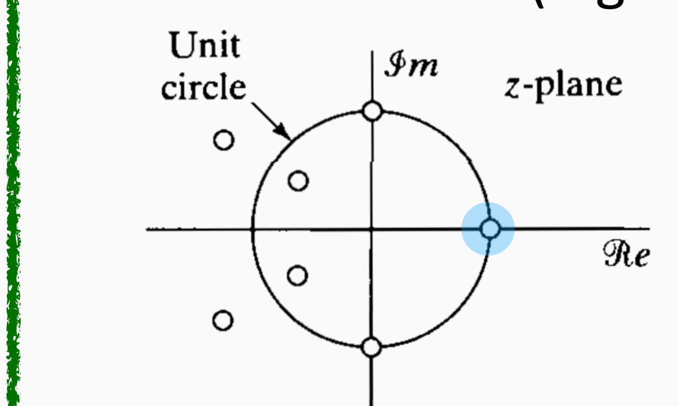




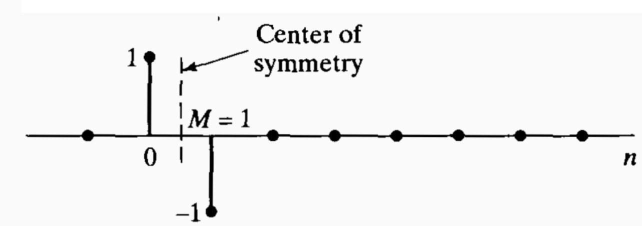


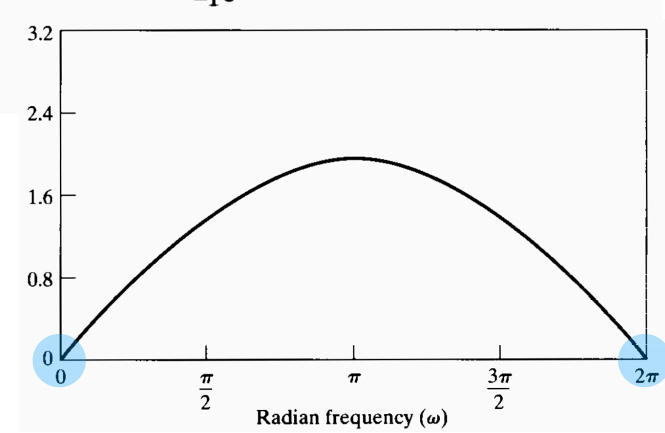
Type 3:
M=even,
h[n]=odd

(high pass filter, differentiator)



Type 4:
M=odd,
h[n]=odd





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