ECE 3640 - Discrete-Time Signals and Systems
Sampling & Reconstruction

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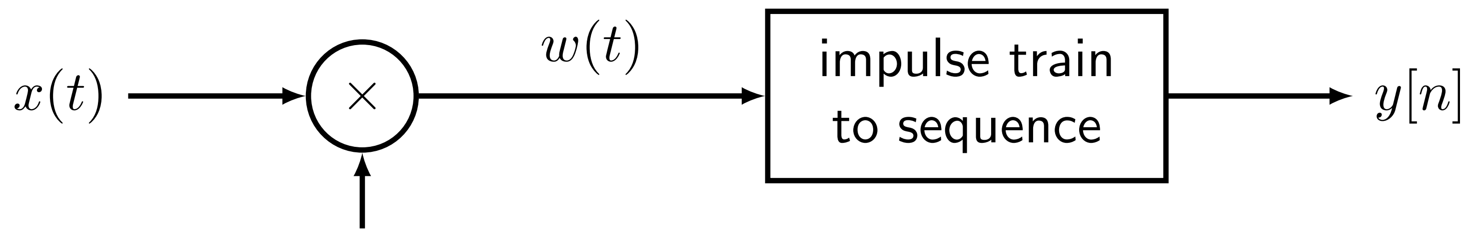
sampling



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- goal of sampling: $y[n] = x(nT)$
- sample rate: $F_s = 1/T$ [samples/second]

$$w(t) = x(t)p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT)$$

CTFT of periodic signals: by example

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{k=-\infty}^{\infty} P_k e^{j2\pi \frac{k}{T} t} \quad \text{expand in CTFS}$$

$$P_k = \frac{1}{T} \int_0^T p(t) e^{-j2\pi \frac{k}{T} t} dt = \frac{1}{T} \int_{0^-}^{T^-} \delta(t) e^{-j2\pi \frac{k}{T} t} dt = \frac{1}{T}$$

Now invoke linearity of CTFT and CTFT pair $e^{j2\pi F_0 t} \leftrightarrow \delta(F - F_0)$:

$$p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j2\pi \frac{k}{T} t}$$

$$P(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(F - \frac{k}{T}\right)$$

math of sampling

$$w(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT)$$

$$W(F) = X(F) * P(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(F - \frac{k}{T}\right) \quad \text{CTFT of } x(t)p(t)$$

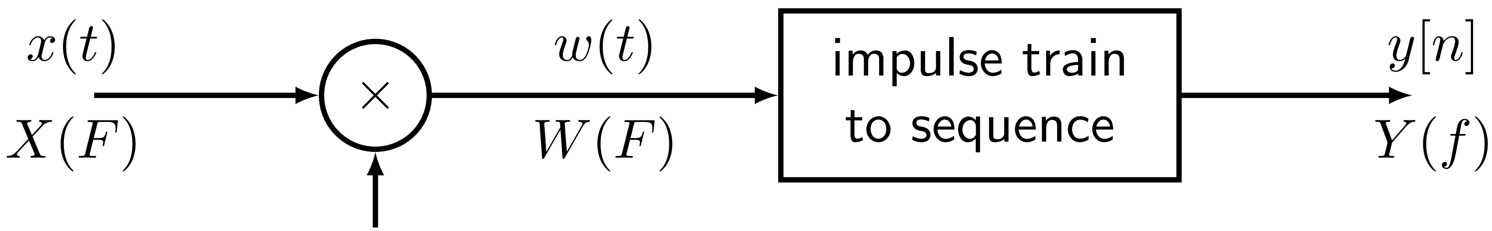
$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT)e^{-j2\pi Ft}dt \quad \text{CTFT of } \sum_{n} y[n]\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} y[n] \int_{-\infty}^{\infty} \delta(t - nT)e^{-j2\pi Ft}dt$$

$$= \sum_{n=-\infty}^{\infty} y[n]e^{-j2\pi FTn} = Y(FT)$$

$$Y(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{f-k}{T}\right)$$

summary

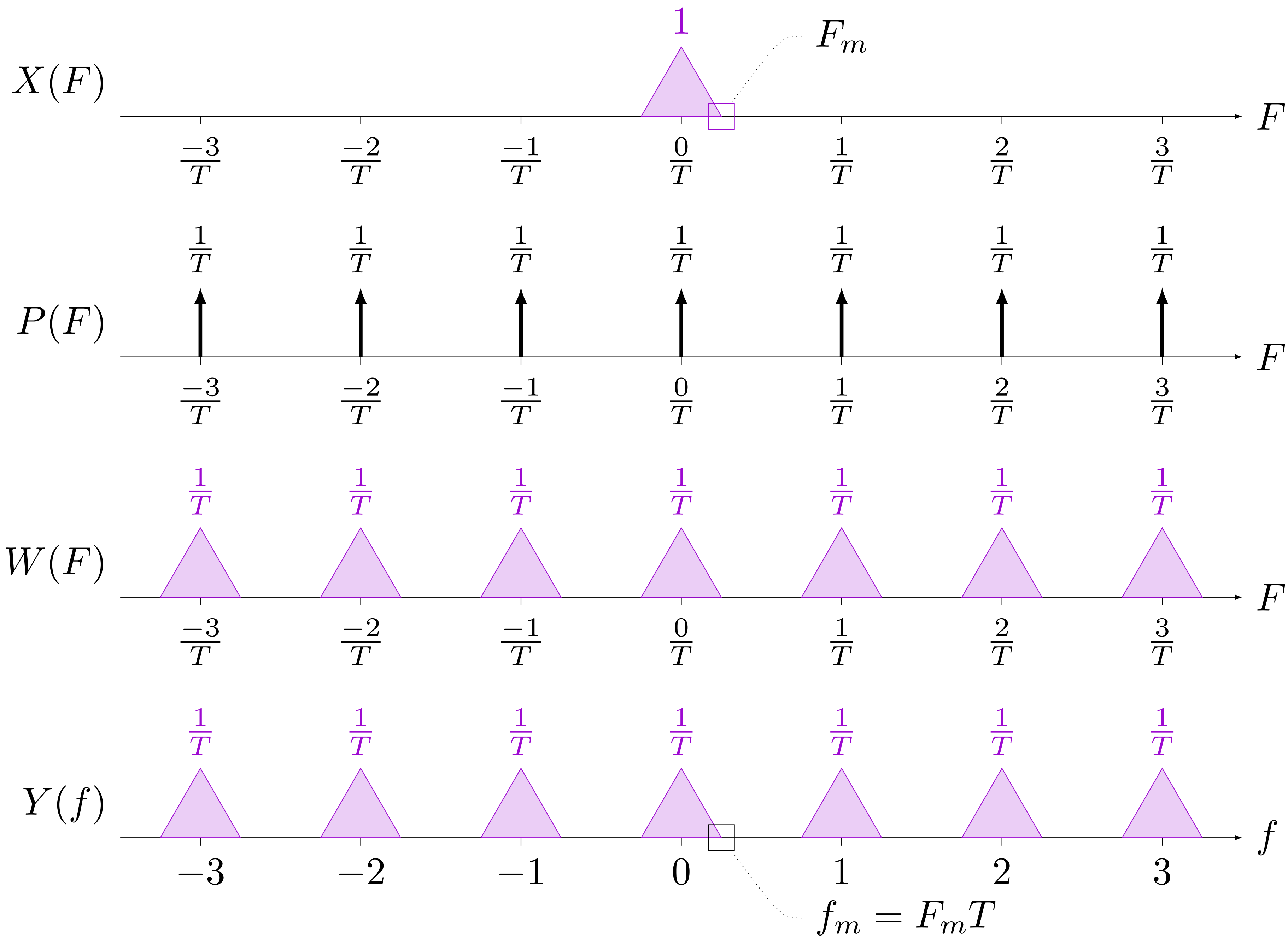


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \longleftrightarrow \quad P(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(F - \frac{k}{T}\right)$$

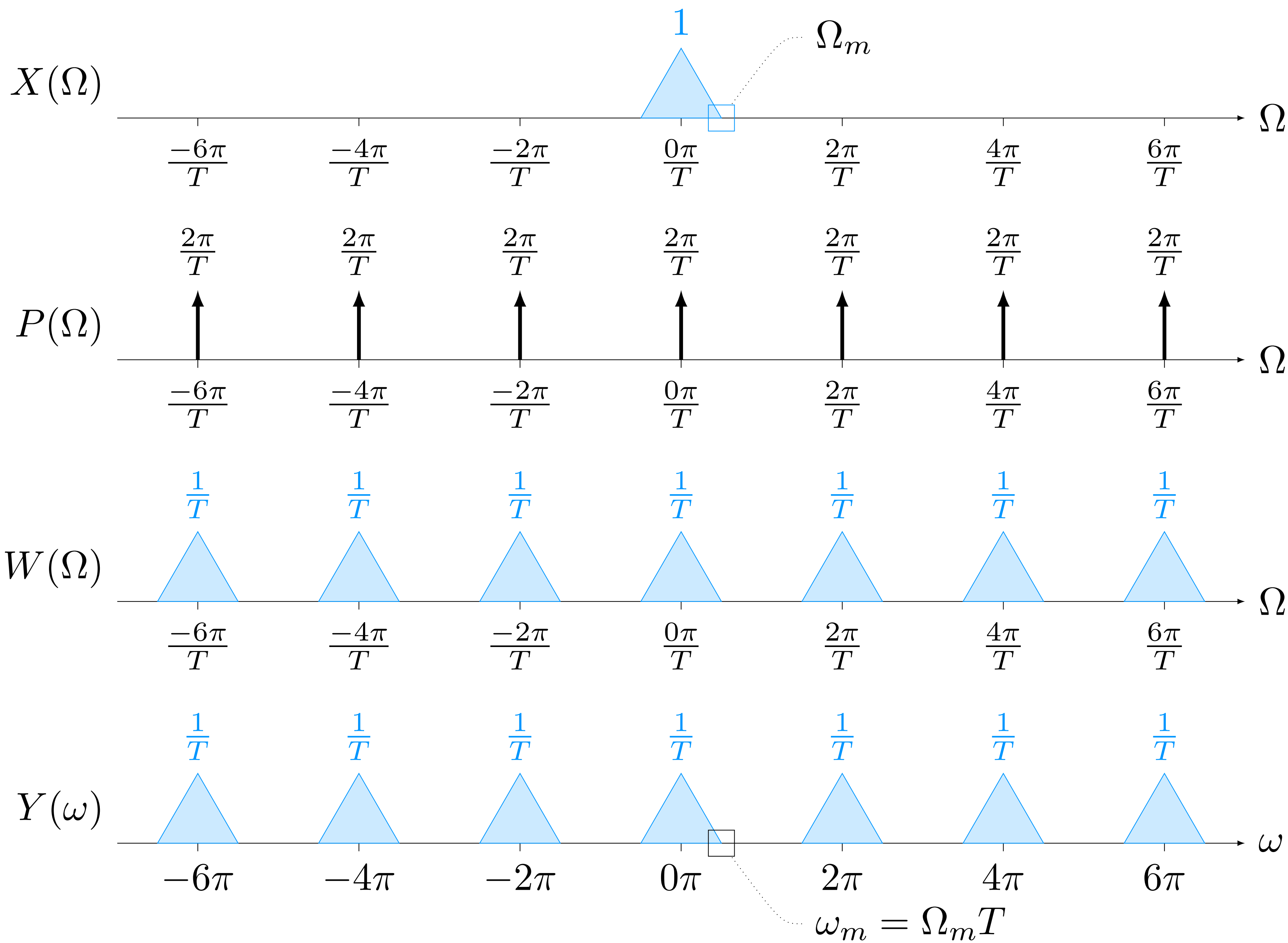
aliasing formula: $$W(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(F - \frac{k}{T}\right)$$

sampling formula: $$Y(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{f - k}{T}\right)$$

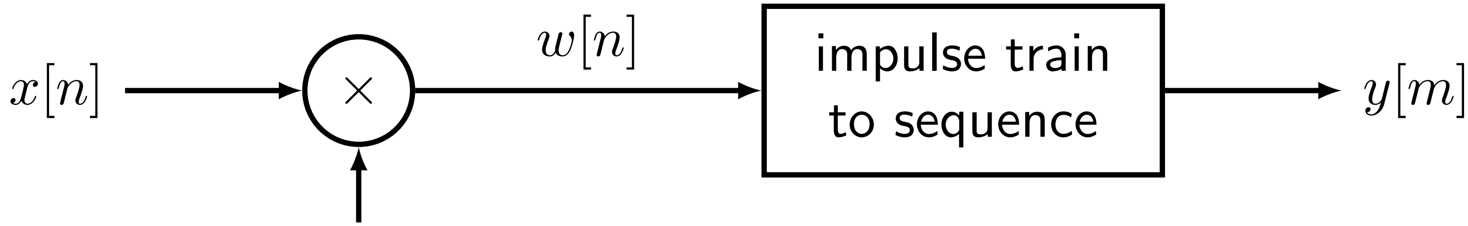
graphical approach



graphical approach



down sampling



$$p[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$

- goal of down sampling: $y[m] = x[mN]$
- sample rate: $f_s = 1/N$ [samples/sample]

$$w[n] = x[n]p[n] = x[n] \sum_{m=-\infty}^{\infty} \delta[n - mT] = \sum_{n=-\infty}^{\infty} x[mN]\delta[n - mN]$$

$$= \sum_{n=-\infty}^{\infty} y[m]\delta[n - mN]$$

DTFT of periodic signals: by example

$$p[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN] = \sum_{k=0}^{N} P_k e^{j2\pi \frac{k}{N} n} \quad \text{expand in DTFS}$$

$$P_k = \frac{1}{N} \sum_{n=0}^{N} p[n] e^{-j2\pi \frac{k}{N} n} = \frac{1}{N} \sum_{n=0}^{N} \delta[n] e^{-j2\pi \frac{k}{N} n} = \frac{1}{N}$$

Now invoke linearity of DTFT and DTFT pair $e^{j2\pi f_0 t} \leftrightarrow \sum_l \delta(f - f_0 - l)$:

$$p[n] = \frac{1}{N} \sum_{k=0}^{N} e^{j2\pi \frac{k}{N} n}$$

$$P(f) = \frac{1}{N} \sum_{k=0}^{N} \sum_{l=-\infty}^{\infty} \delta\left(f - \frac{k}{N} - l\right)$$

math of down sampling

$$w[n] = x[n]p[n] = \sum_{m=-\infty}^{\infty} y[m]\delta[n - mN]$$

$$W(f) = X(f) \circledast P(f) = \frac{1}{N}\sum_{k=0}^{N} X\left(f - \frac{k}{N}\right)$$ DTFT of $x[n]p[n]$

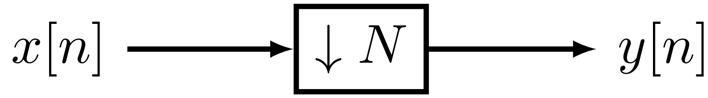
$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} y[m]\delta[n - mN]e^{-j2\pi f n}dt$$ DTFT of $\sum_{m} y[m]\delta[n - mN]$

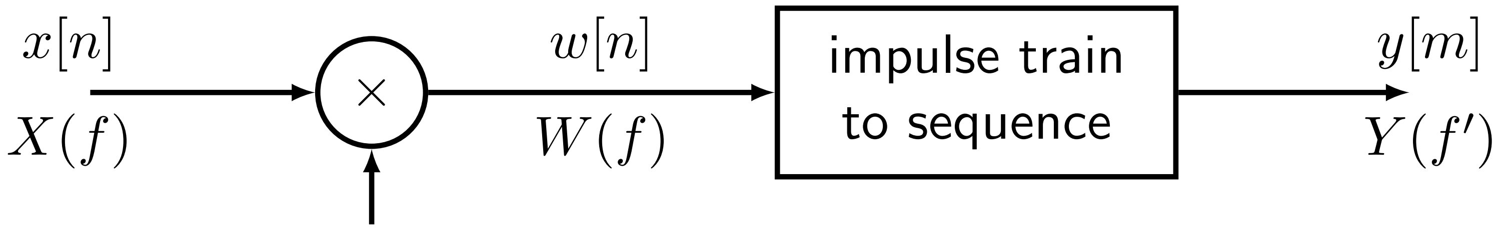
$$= \sum_{m=-\infty}^{\infty} y[m]\sum_{n=-\infty}^{\infty} \delta[n - mN]e^{-j2\pi f n}$$

$$= \sum_{m=-\infty}^{\infty} y[m]e^{-j2\pi f N m} = Y(fN)$$

$$Y(f') = \frac{1}{N}\sum_{k=0}^{N} X\left(\frac{f' - k}{N}\right)$$

summary



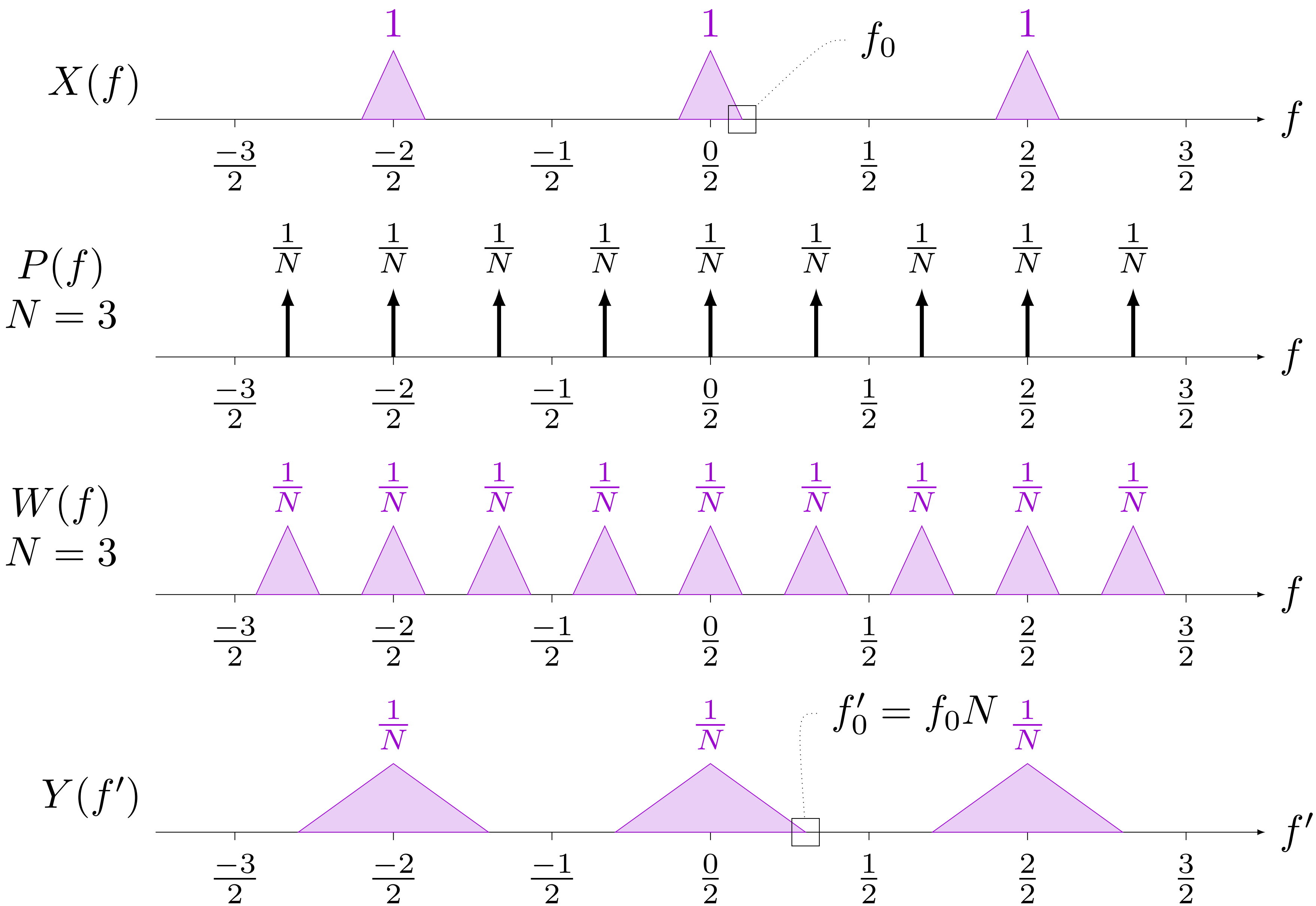


$$p[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN] \quad \leftrightarrow \quad P(f) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{m}{N} - k\right)$$

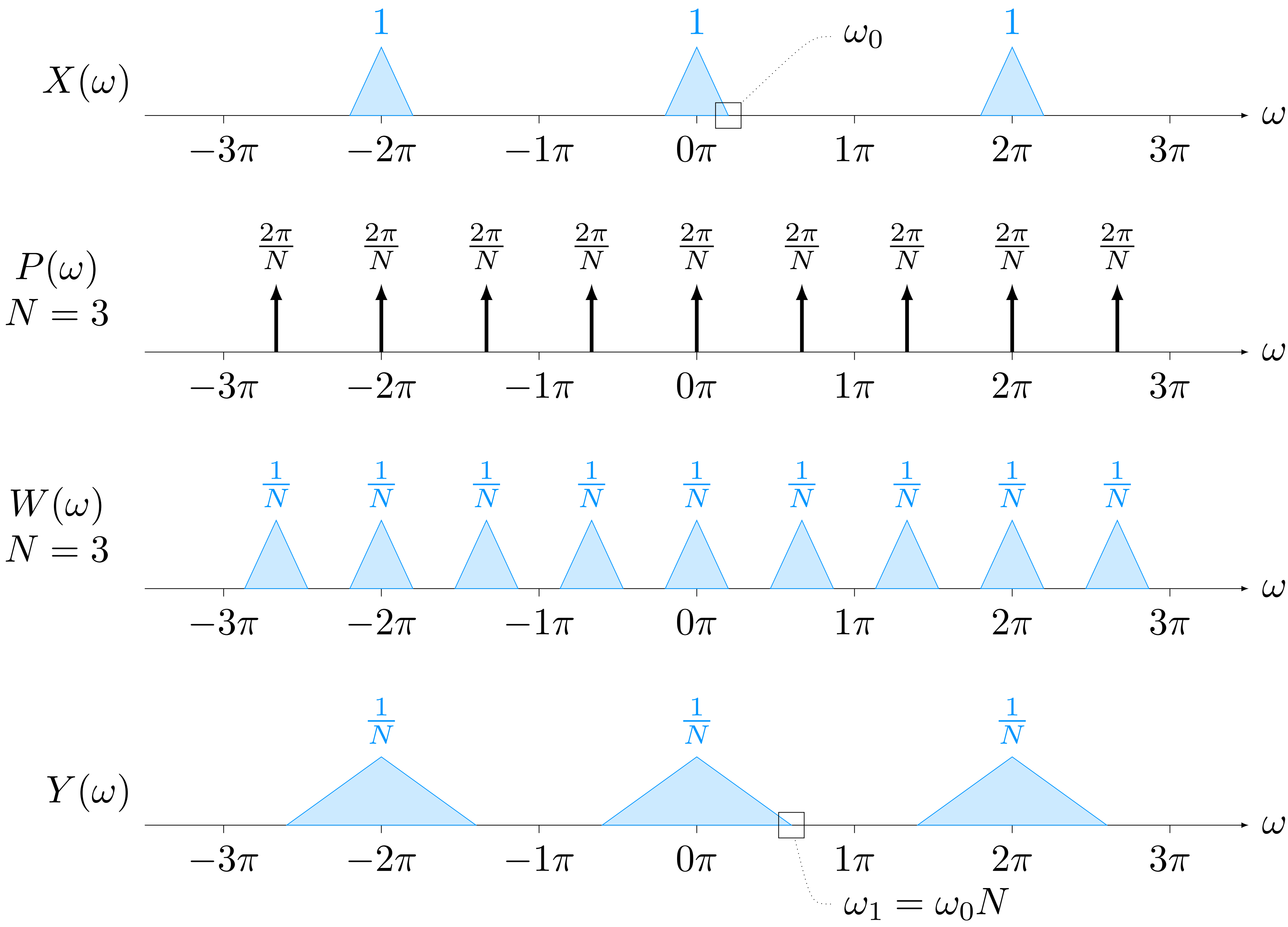
aliasing formula: $$W(f) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(f - \frac{k}{N}\right)$$

down sampling formula: $$Y(f') = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{f' - k}{N}\right)$$

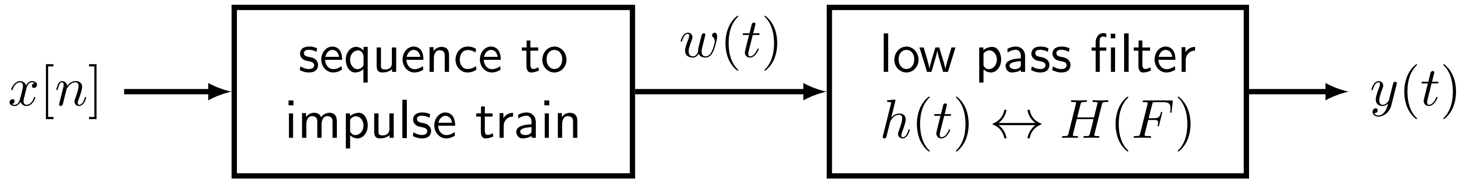
graphical approach



graphical approach



reconstruction



- goal of reconstruction: synthesize low-pass signal $y(t)$ that agrees with $x[n]$ at the sample times, $y(nT) = x[n]$

- reconstruction rate: $F_s = 1/T$ [samples/second]

$$w(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

$$W(F) = \int_{-\infty}^{\infty} w(t)e^{-j2\pi Ft}dt = \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT)e^{-j2\pi Ft}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi FTn} = X(FT) \qquad \text{(scale frequency axis)}$$

reconstruction

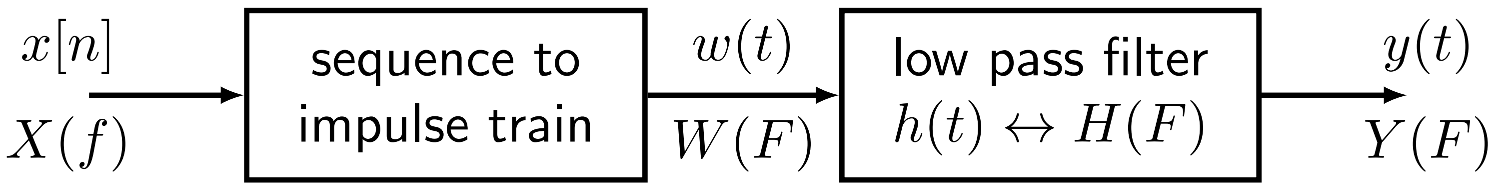
- apply low pass filter to remove images
- assume ideal low pass filter

$$H(F) = \begin{cases} T, & |F| \leq \frac{1}{2T} = \frac{F_s}{2} \\ 0, & \text{otherwise} \end{cases} \quad \longleftrightarrow \quad h(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}}$$

$$y(t) = h(t) * w(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT)$$

$$Y(F) = \begin{cases} X(FT), & |F| \leq \frac{1}{2T} = \frac{F_s}{2}, \\ 0, & \text{otherwise} \end{cases}$$

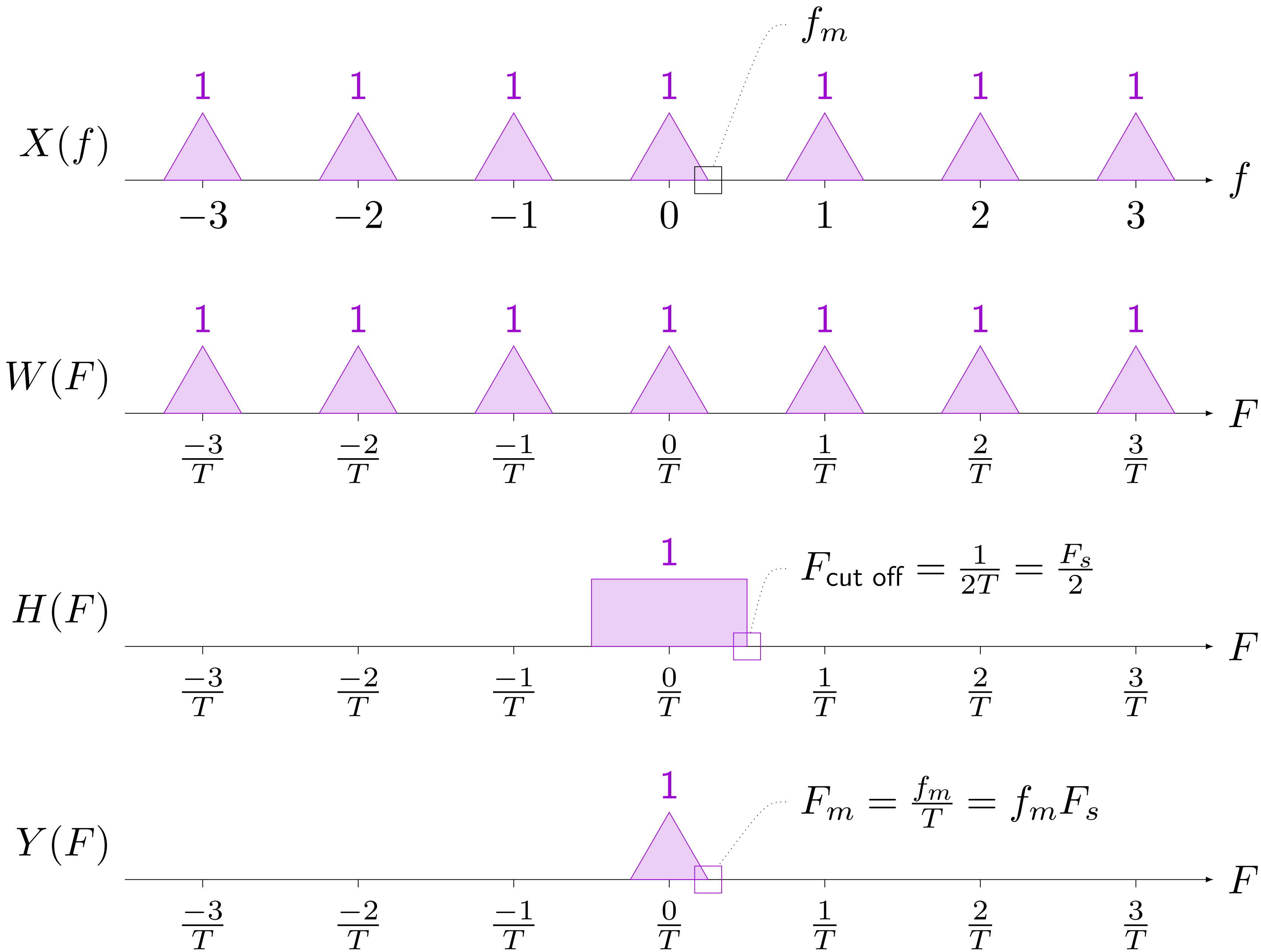
reconstruction summary



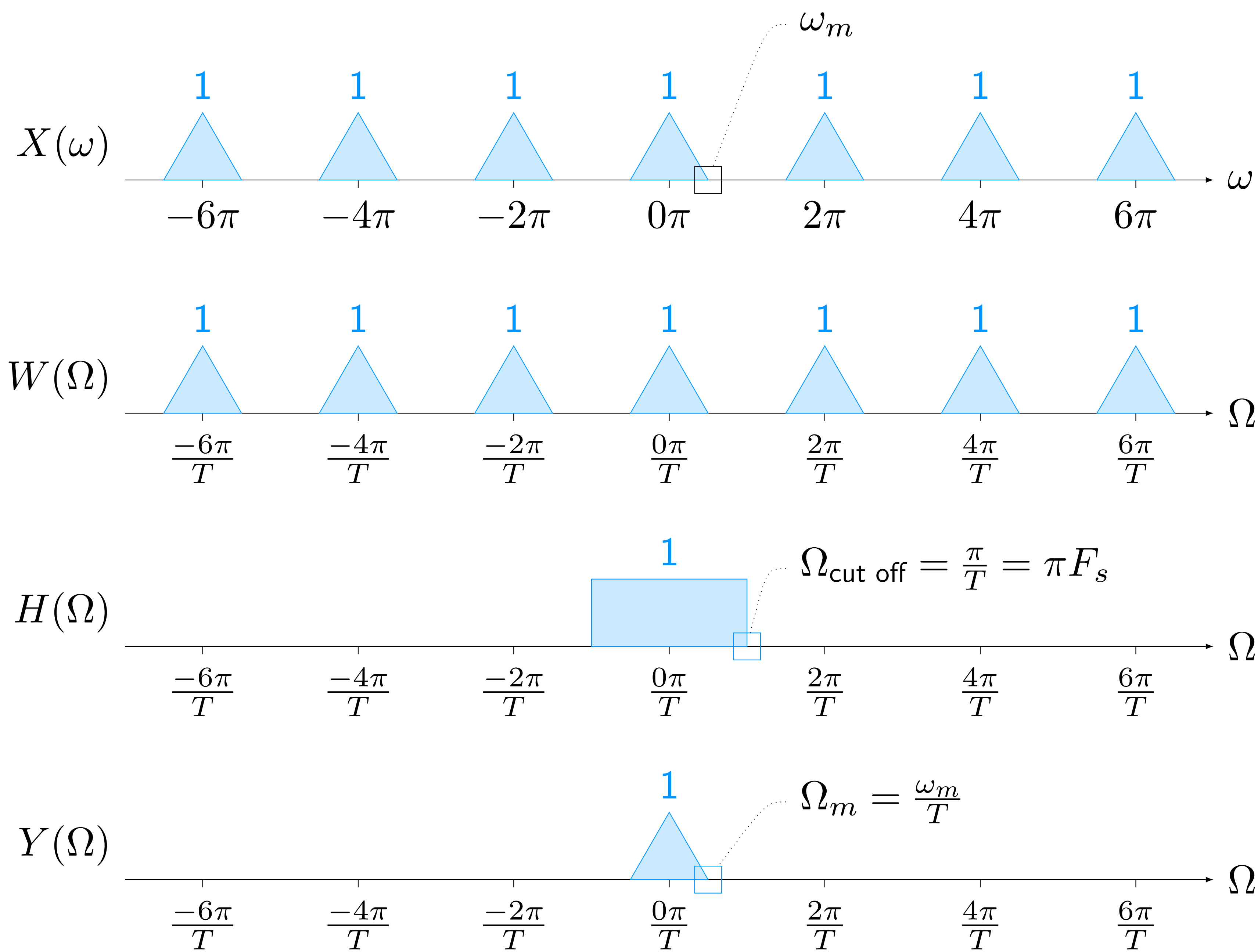
imaging formula: $W(F) = X(FT) = X\left(\frac{F}{F_s}\right)$

LPF to remove images: $Y(F) = \begin{cases} X(FT) = X\left(\frac{F}{F_s}\right), & |F| \leq \frac{1}{2T} = \frac{F_s}{2}, \\ 0, & \text{otherwise} \end{cases}$

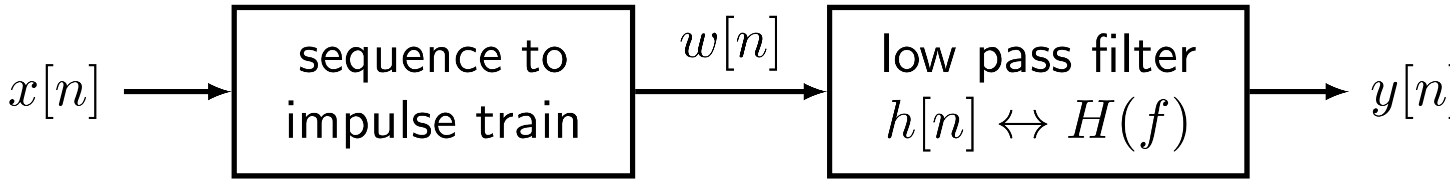
graphical approach to reconstruction



graphical approach to reconstruction



up sampling



- goal of up sampling: synthesize low-pass signal $y[n]$ that agrees with $x[n]$ at the sample times, $y[nN] = x[n]$

- reconstruction rate: $f_s = 1/N$ [samples/sample]

$$w[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n - mN]$$

$$W(f) = \sum_{n=-\infty}^{\infty} w[n]e^{-j2\pi fn} = \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} \delta[n - mN]e^{-j2\pi fn}$$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j2\pi fNm} = X(fN) \qquad \text{(scale frequency axis)}$$

up sampling

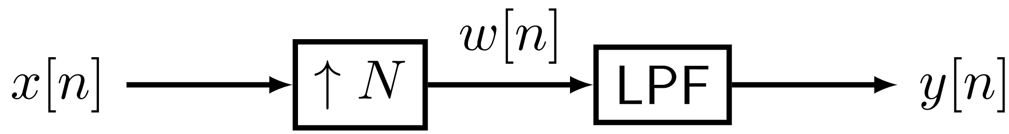
- apply low pass filter to remove images
- assume ideal low pass filter

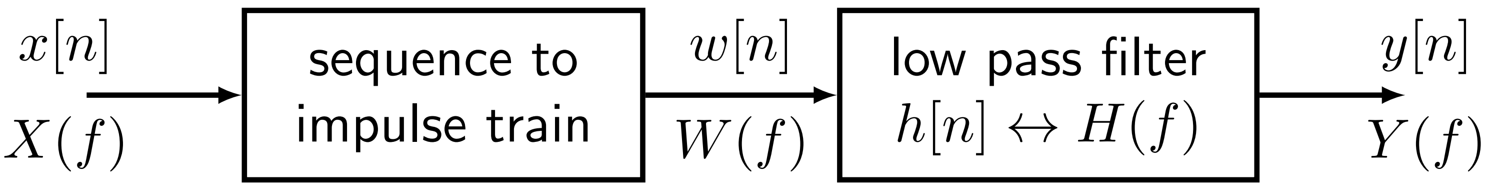
$$H(f) = \begin{cases} N, & |f| \leq \frac{1}{2N} = \frac{f_s}{2} \\ 0, & \text{otherwise} \end{cases} \quad \longleftrightarrow \quad h[n] = \frac{\sin\left(\frac{\pi n}{N}\right)}{\frac{\pi n}{N}}$$

$$y[n] = h[n] * w[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - mN]$$

$$Y(f) = \begin{cases} X(fN), & |f| \leq \frac{1}{2N} = \frac{f_s}{2}, \\ 0, & \text{otherwise} \end{cases}$$

up sampling summary

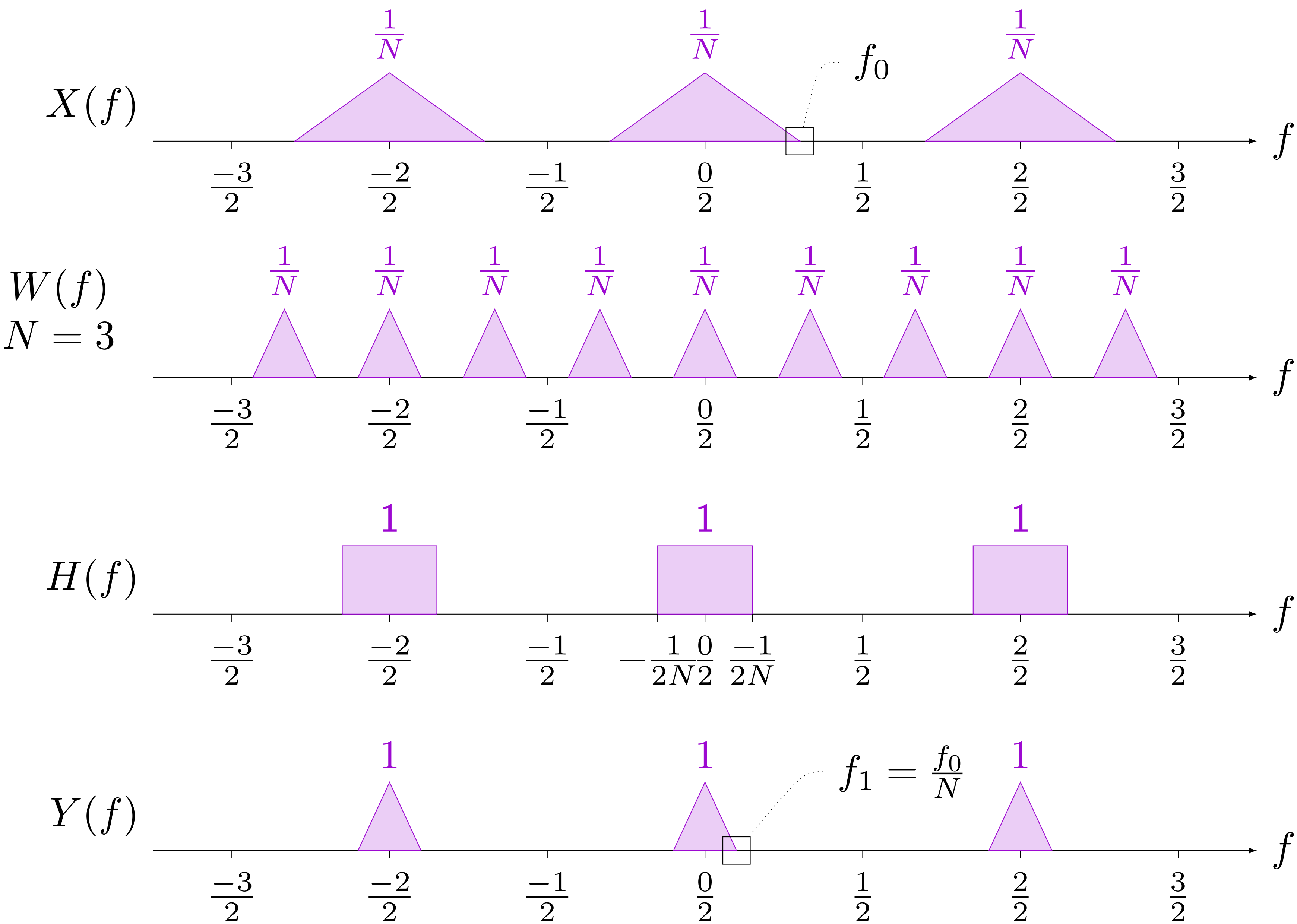




imaging formula: $W(f) = X(fN) = X\left(\frac{f}{f_s}\right)$

LPF to remove images: $Y(f) = \begin{cases} X(fN) = X\left(\frac{f}{f_s}\right), & |f| \leq \frac{1}{2N} = \frac{f_s}{2}, \\ 0, & \text{otherwise} \end{cases}$

graphical approach to up sampling



graphical approach to upsampling