ECE 3640 - Discrete-Time Signals and Systems
Proof of Geometric Series Formulas

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formulas

finite limits:

$$\sum_{n=M}^{N-1} a^n = \begin{cases} N - M, & a = 1 \\ \\ \dfrac{a^M - a^N}{1-a}, & a \neq 1 \end{cases}$$

infinite upper limit:

$$\sum_{n=M}^{\infty} a^n = \frac{a^M}{1-a}, \qquad |a| < 1$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \qquad |a| < 1$$

finite limits $a = 1$

$$\sum_{n=M}^{N-1} 1 = \underbrace{1 + 1 + \cdots + 1}_{N-M \text{ ones}} = N - M$$

example: $M = 0, N = 1$

$$\sum_{n=0}^{1-1=0} 1 = 1 = N - M$$

example: $M = 0, N = 2$

$$\sum_{n=0}^{2-1=1} 1 = 1 + 1 = 2 = N - M$$

example: $M = 20, N = 23$

$$\sum_{n=20}^{23-1=22} 1 = 1 + 1 + 1 = 3 = N - M$$

finite limits $a \neq 1$

$$\sum_{n=M}^{N-1} a^n = a^M + a^{M+1} + \cdots + a^{N-1}$$

$$a \sum_{n=M}^{N-1} a^n = \qquad a^{M+1} + \cdots + a^{N-1} + a^N$$

$$\sum_{n=M}^{N-1} a^n - a \sum_{n=M}^{N-1} a^n = a^M + \qquad\qquad - a^N$$

$$(1-a) \sum_{n=M}^{N-1} a^n = a^M - a^N$$

$$\sum_{n=M}^{N-1} a^n = \frac{a^M - a^N}{1-a} = \frac{a^N - a^M}{a-1}$$

infinite upper limit

Let $a \in \mathbb{C}$, then $a = re^{j\omega}$. Consider the behavior of $a^n = r^n e^{j\omega n}$, where $r = |a|$

$$a^n = r^n e^{j\omega n} = \begin{cases} r < 1, & \text{decaying oscillation} \\ r = 1, & \text{everlasting oscillation} \\ r > 1, & \text{unbounded oscillation} \end{cases}$$

Now take the limit,

$$\lim_{n\to\infty} a^n = r^n e^{j\omega n} = \begin{cases} 0, & r < 1 \\ ?, & r = 1 \\ \infty, & r > 1 \end{cases}$$

$$\sum_{n=M}^{\infty} a^n = \lim_{N\to\infty} \sum_{n=M}^{N-1} a^n = \lim_{N\to\infty} \frac{a^M - a^N}{1-a} = \frac{a^M}{1-a}, \quad \text{provided} \quad |a| < 1$$

$$\text{Let } M = 0\text{:} \quad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

formula summary

finite limits:

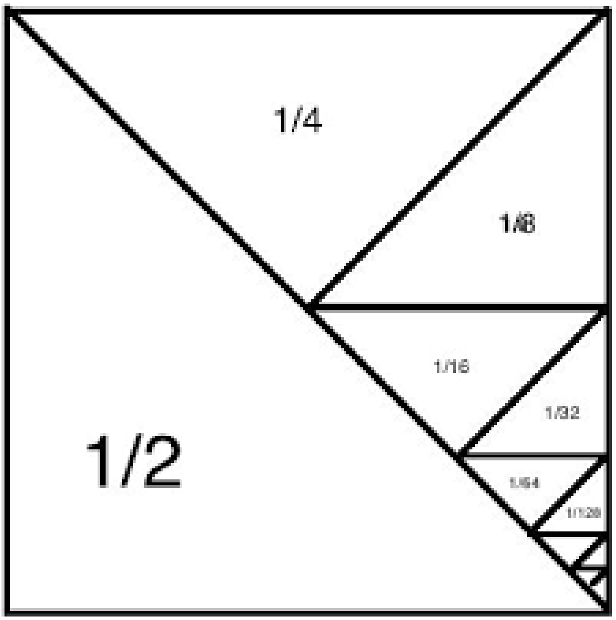
$$\sum_{n=M}^{N-1} a^n = \begin{cases} N - M, & a = 1 \\ \dfrac{a^M - a^N}{1-a}, & a \neq 1 \end{cases}$$

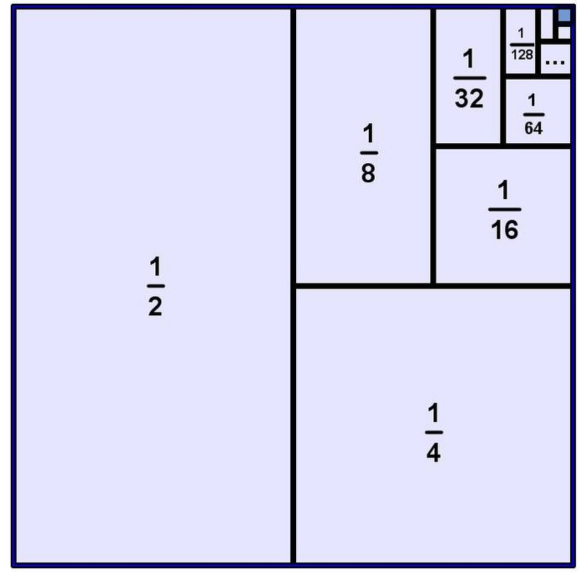
infinite upper limit:

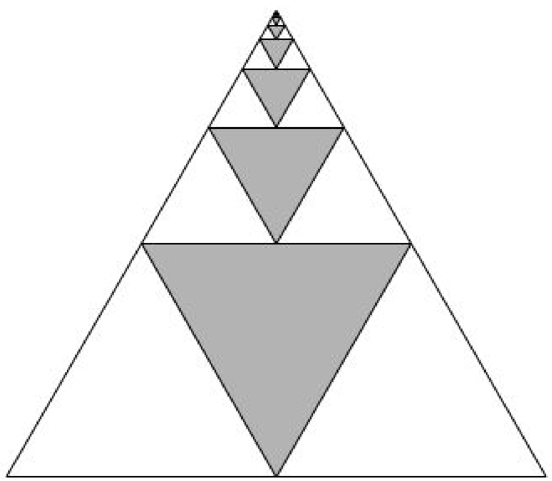
$$\sum_{n=M}^{\infty} a^n = \frac{a^M}{1-a}, \qquad |a| < 1$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \qquad |a| < 1$$

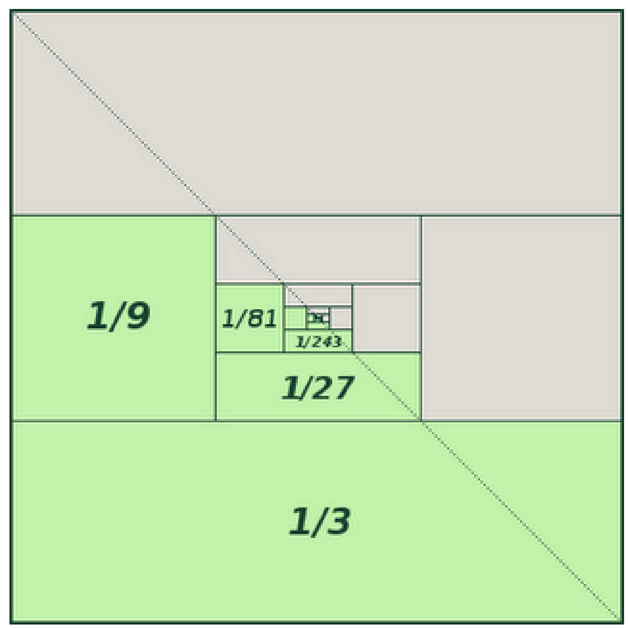
proof without words







$$\frac{1}{3} = \sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} \dots$$



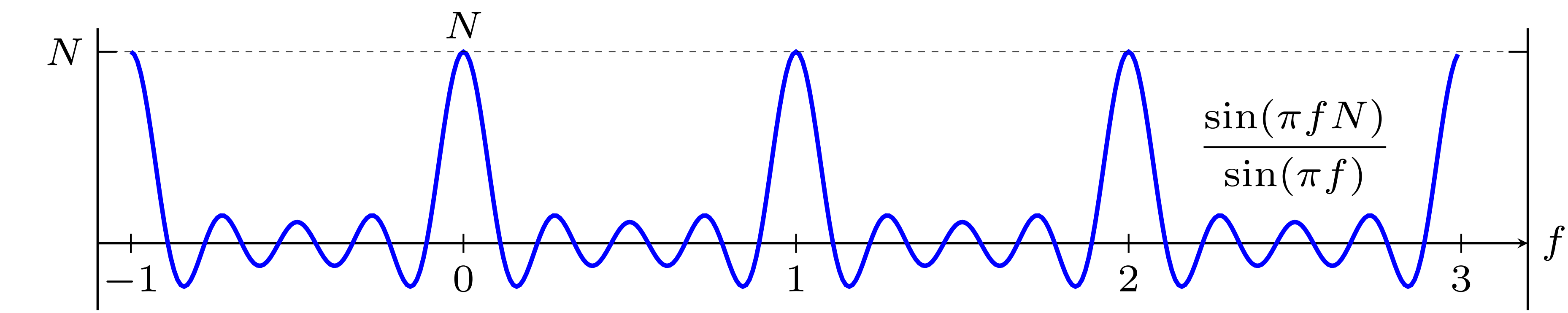
sum of exponential signals: symmetric limits

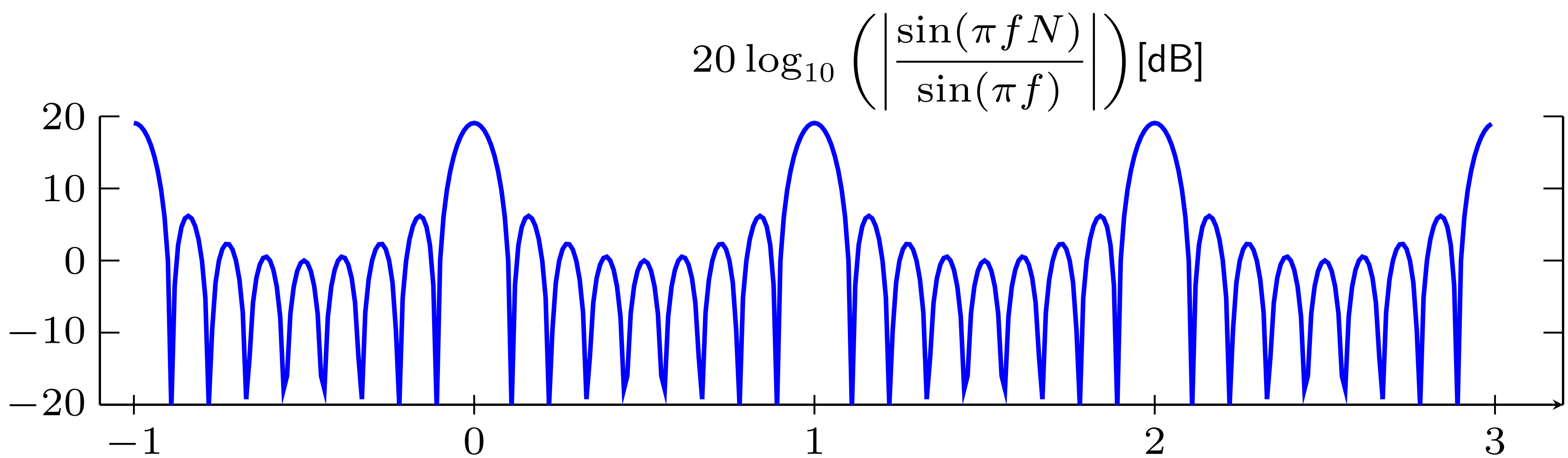
$$
\begin{aligned}
\sum_{n=-N}^{N} e^{-j2\pi f n} &= \frac{e^{-j2\pi f(-N)} - e^{-j2\pi f(N+1)}}{1 - e^{-j2\pi f}} \\
&= \frac{e^{j2\pi f N} - e^{-j2\pi f(N+1)}}{1 - e^{-j2\pi f}} \\
&= \frac{e^{j2\pi f(N+\frac{1}{2})} - e^{-j2\pi f(N+\frac{1}{2})}}{e^{j\pi f} - e^{-j\pi f}} \; \frac{e^{-j\pi f}}{e^{-j\pi f}} \\
&= \frac{e^{j2\pi f(N+\frac{1}{2})} - e^{-j2\pi f(N+\frac{1}{2})}}{2j} \; \frac{2j}{e^{j\pi f} - e^{-j\pi f}} \\
&= \frac{\sin(2\pi f(N+\frac{1}{2}))}{\sin(\pi f)} \\
&= \frac{\sin(\pi f(2N+1))}{\sin(\pi f)}
\end{aligned}
$$

sum of exponential signals: non-symmetric limits

$$
\begin{aligned}
\sum_{n=0}^{N-1} e^{-j2\pi f n} &= \frac{1 - e^{-j2\pi f N}}{1 - e^{-j2\pi f}} \\
&= \frac{e^{j\pi f N} - e^{-j\pi f N}}{e^{j\pi f} - e^{-j\pi f}} \; \frac{e^{-j\pi f N}}{e^{-j\pi f}} \\
&= \frac{e^{j\pi f N} - e^{-j\pi f N}}{2j} \; \frac{2j}{e^{j\pi f} - e^{-j\pi f}} \; \frac{e^{-j\pi f N}}{e^{-j\pi f}} \\
&= \frac{\sin(\pi f N)}{\sin(\pi f)} e^{-j\pi f (N-1)}
\end{aligned}
$$

periodic sinc plots ($N = 9$)





summary

Summing complex exponential leads to periodic sinc function.

Periodic sinc an imporant function for understanding digital signal processing theory.

$$\sum_{n=a}^{b-1} e^{\pm j 2\pi f n} = \frac{\sin(\pi f (b-a))}{\sin(\pi f)} e^{\pm j \pi f (b+a-1)}$$

periodic sinc in Matlab

How can $\frac{\sin(\pi f N)}{\sin(\pi f)}$ be plotted in Matlab?

The following function is defined in Matlab

$$\texttt{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

If you enter `type sinc` at the Matlab command prompt, you can see how the Mathworks coded their `sinc` function.

```
function y=sinc(x)
% Mathworks sinc function
i=find(x==0);
x(i)= 1;
y = sin(pi*x)./(pi*x);
y(i) = 1;
```

We can copy the Mathwork's style to make a new periodic sinc function.

periodic sinc function

```
1 function y=sincperiodic(x,N)
2 i=find(mod(x,1)==0); % 0/0 occurs whenever x is an integer
3 x(i) = 0.5;
4 y = sin(pi*x*N)./sin(pi*x);
5 y(i) = N;
```

- Line 2 finds all places in the input where the output should be $\frac{\sin(\pi x N)}{\sin(\pi x)} = \frac{0}{0} = N$
- Line 3 resets the input at these places to avoid division by zero
- Line 4 evaluates the function
- Line 5 cleans up the $\frac{0}{0}$ places in the output by setting these to N

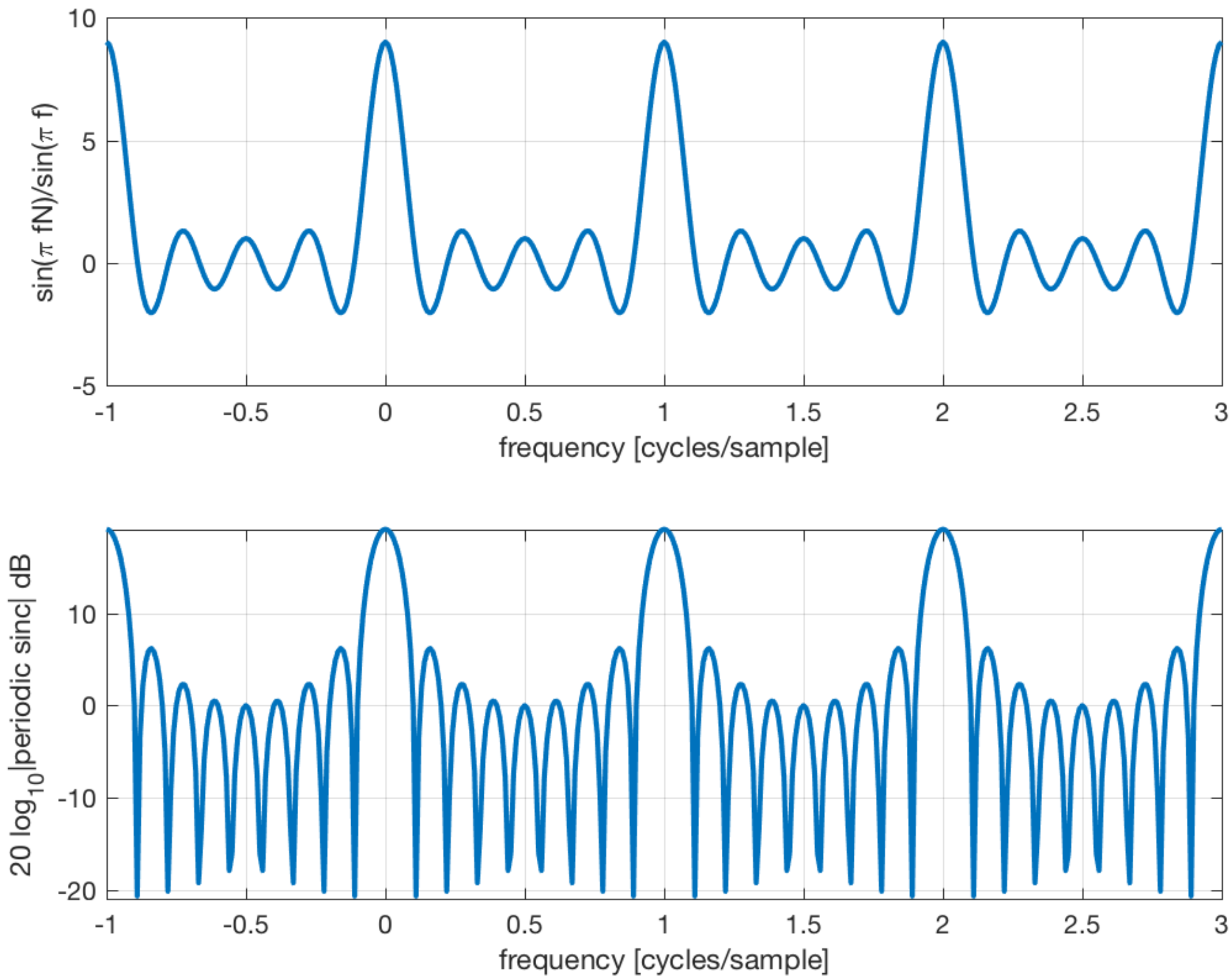
Matlab code

```
N=9
f=[-1:0.01:3]
S=sincperiodic(f,N)
subplot(211);
plot(f,S,'LineWidth',2);
grid on;
xlabel('frequency [cycles/sample]');
ylabel('sin(\pi fN)/sin(\pi f)');
subplot(212);
plot(f,20*log10(abs(S)),'LineWidth',2);
grid on;
ylim(20*log10(N) + [-40 0]);
xlabel('frequency [cycles/sample]');
ylabel('20 log_{10}|periodic sinc| dB');
orient landscape;
print -dpng sincperiodicplot.png
```

Run these commands one at a time to see what each one does.

See plot on next page.

Matlab plot


10
5
0
-5
sin(π fN)/sin(π f)
-1
-0.5
0
0.5
1
1.5
2
2.5
3
frequency [cycles/sample]
10
0
-10
-20
20 log10|periodic sinc| dB
-1
-0.5
0
0.5
1
1.5
2
2.5
3
frequency [cycles/sample]

assignment

Do the following:

- perform the sums below (show all intermediate steps)
- put answers in periodic sinc function form (show your work)
- make linear and logarithmic plots in Matlab

$$\sum_{n \in S} e^{-j2\pi f n}$$

1. $S = [-5, 5] = \{n \in \mathbb{Z}, -5 \leq n \leq 5\}$
2. $S = [0, 10]$ (plot real and imaginary parts on same axis using Matlab's `hold on` and `hold off` commands)
3. $S = [-10, -5] \cup [5, 10]$