ECE 3640 - Discrete-Time Signals and Systems

Integral of Exponential Function

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calculus of e^t

$$\frac{d}{dt}e^t = e^t \qquad \int e^t dt = e^t + c$$

$$\frac{d}{dt}e^{at} = ae^{at} \qquad \int e^{at} dt = \frac{e^{at}}{a} + c$$

$$\int_u^v e^{at} dt = \left.\frac{e^{at}}{a}\right|_u^v = \frac{e^{av} - e^{au}}{a}$$

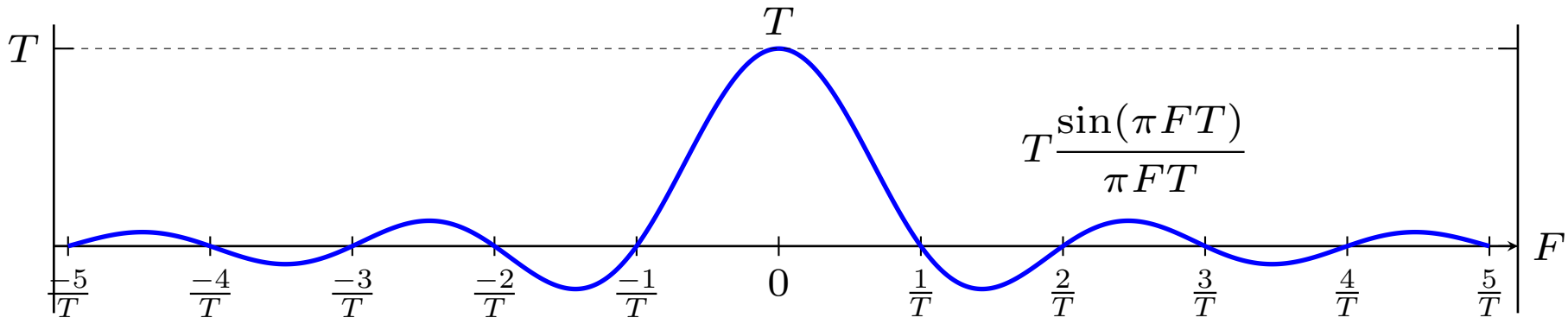
integral of exponential signals

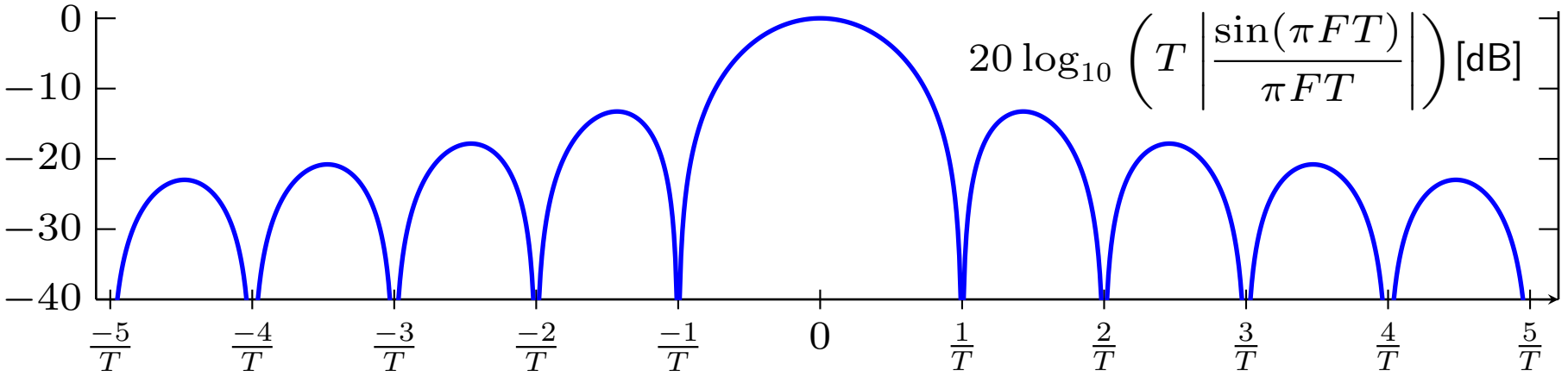
$$\int_u^v e^{at} dt = \frac{e^{at}}{a}\bigg|_u^v = \frac{e^{av} - e^{au}}{a}$$

Example: $a = j2\pi F \neq 0, v = T/2, u = -T/2,$

$$\begin{aligned}
\int_{-T/2}^{T/2} e^{j2\pi Ft} dt &= \frac{e^{j2\pi F\frac{T}{2}} - e^{-j2\pi F\frac{T}{2}}}{j2\pi F} \\
&= \frac{1}{\pi F}\left(\frac{e^{j\pi FT} - e^{-j\pi FT}}{j2}\right) \\
&= \frac{\sin(\pi FT)}{\pi F} \\
&= T\frac{\sin(\pi FT)}{\pi FT} \quad \text{(sinc function)}
\end{aligned}$$

sinc plots


T
T
$T\frac{\sin(\pi FT)}{\pi FT}$
F
$\frac{-5}{T}$ $\frac{-4}{T}$ $\frac{-3}{T}$ $\frac{-2}{T}$ $\frac{-1}{T}$ 0 $\frac{1}{T}$ $\frac{2}{T}$ $\frac{3}{T}$ $\frac{4}{T}$ $\frac{5}{T}$


0
−10
−20
−30
−40
$20\log_{10}\left(T\left|\frac{\sin(\pi FT)}{\pi FT}\right|\right)$ [dB]
$\frac{-5}{T}$ $\frac{-4}{T}$ $\frac{-3}{T}$ $\frac{-2}{T}$ $\frac{-1}{T}$ 0 $\frac{1}{T}$ $\frac{2}{T}$ $\frac{3}{T}$ $\frac{4}{T}$ $\frac{5}{T}$

another example

Symmetric exponents identity (check this by multiplication):

$$\int_u^v e^{at}dt = \frac{e^{at}}{a}\bigg|_u^v = \frac{e^{av}-e^{au}}{a} = \frac{e^{a\left(\frac{v-u}{2}\right)}-e^{-a\left(\frac{v-u}{2}\right)}}{a}e^{a\left(\frac{v+u}{2}\right)}$$

(factor out $e^{(\text{average of exponents})}$)

Example: $a = j2\pi F \neq 0, v = T, u = 0,$

$$\begin{aligned}\int_0^T e^{j2\pi Ft}dt = \frac{e^{j2\pi FT}-1}{j2\pi F} &= \frac{e^{j2\pi F\frac{(T-0)}{2}}-e^{-j2\pi F\frac{(T-0)}{2}}}{j2\pi F}e^{j2\pi F\frac{(T+0)}{2}} \\ &= \frac{1}{\pi F}\left(\frac{e^{j\pi FT}-e^{-j\pi FT}}{j2}\right)e^{j\pi FT} \\ &= T\frac{\sin(\pi FT)}{\pi FT}e^{j\pi FT}\end{aligned}$$

summary

Integrating complex exponential leads to sinc function.

Sinc an important function for understanding digital signal processing theory.

$$\int_a^b e^{\pm j2\pi uv} dv = \frac{\sin(\pi u(b-a))}{\pi u} e^{\pm j\pi u(b+a)}$$

`sinc` in Matlab

How can $\frac{\sin(\pi FT)}{\pi F}$ be plotted in Matlab?

The following function is defined in Matlab

$$\texttt{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Note that π is included in the definition.

Therefore to use Matlab's `sinc` function, we have to transform our function into $\frac{\sin(\pi x)}{\pi x}$ form:

$$\frac{\sin(\pi FT)}{\pi F} = T\frac{\sin(\pi FT)}{\pi FT} = T\,\texttt{sinc}(FT)$$

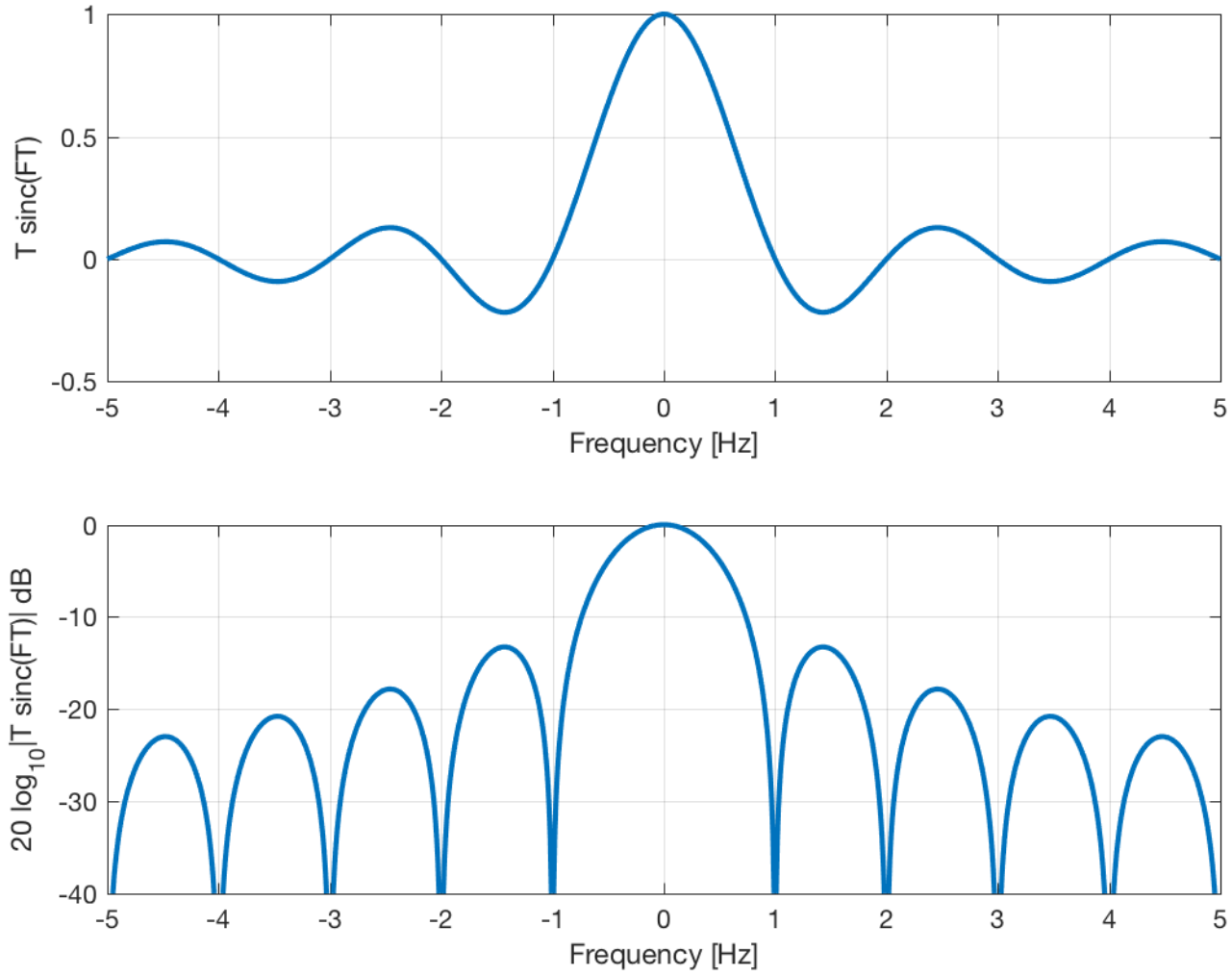
Matlab code

```
T=1
F=[-5:0.01:5]
S=T*sinc(F*T)
subplot(211);
plot(F,S,'LineWidth',2);
grid on;
xlabel('Frequency [Hz]');
ylabel('T sinc(FT)');
subplot(212);
plot(F,20*log10(abs(S)),'LineWidth',2);
grid on;
ylim([-40 0]);
xlabel('Frequency [Hz]');
ylabel('20 log_{10}|T sinc(FT)| dB');
orient landscape;
print -dpng sincplot.png
```

Run these commands one at a time to see what each one does.

See plot on next page.

Matlab plot


T sinc(FT)
1
0.5
0
-0.5
-5
-4
-3
-2
-1
0
1
2
3
4
5
Frequency [Hz]
20 log10|T sinc(FT)| dB
0
-10
-20
-30
-40
-5
-4
-3
-2
-1
0
1
2
3
4
5
Frequency [Hz]

assignment

Do the following:

- perform the integrations below (show all intermediate steps)
- put answers in sinc function form (show your work)
- make linear and logarithmic plots in Matlab

$$\int_S e^{-j2\pi Ft} dt$$

1. $S = [-5, 5] = \{t \in \mathbb{R}, -5 \leq t \leq 5\}$
2. $S = [0, 10]$ (plot real and imaginary parts on same axis using Matlab's `hold on` and `hold off` commands)
3. $S = [-10, -5] \cup [5, 10]$ (hint: union of two intervals)