ECE 3640 - Discrete-Time Signals and Systems
The Discrete-Time Fourier Transform

Jake Gunther

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UtahStateUniversity

Department of Electrical & Computer Engineering

CT complex exponential: property 1

$$x(t) = \exp(j2\pi F_0 t)$$

- periodic for all frequencies $-\infty < F_0 < \infty$
- period $T_0 = 1/F_0$

$$\begin{aligned} x(t + T_0) &= \exp(j2\pi F_0(t + T_0)) \\ &= \exp(j2\pi F_0 t)\underbrace{\exp(j2\pi F_0 T_0)}_{1} \\ &= \exp(j2\pi F_0 t) \\ &= x(t) \end{aligned}$$

CT complex exponential: property 2

- two complex expoentials with different frequencies are different
- let $F_1 \neq F_2$, then

$$x_1(t) = \exp(j2\pi F_1 t) \neq \exp(j2\pi F_2 t) = x_2(t) \quad \text{for all} \quad t$$

- they may be equal at some times, but are not equal everywhere

CT complex exponential: property 3

$$x(t) = \exp(j2\pi F_0 t)$$

- the rate of oscillation increases indefinitely as $F_0 \to \infty$ or as $T_0 \to 0$

CT complex exponential: property 4

- a time shift is equivalent to a phase shift

$$x(t-\tau) = \exp(j2\pi F_0(t-\tau)) = \exp(j[2\pi F_0 t - \varphi]), \quad \varphi = 2\pi F_0 \tau$$

- for every phase shift φ there exists a time shift τ

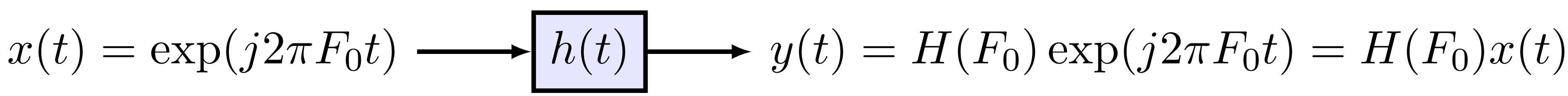
CT complex exponential: property 5

- infinite number of harmonically related and orthogonal complex exponentials
- let $s_k(t) = \exp(j2\pi k F_0 t), \quad k = \cdots, -2, -1, 0, 1, 2, \cdots$

$$
\begin{aligned}
\int_{-T_0/2}^{T_0/2} s_k(t) s_m^*(t) dt &= \int_{-T_0/2}^{T_0/2} \exp(j2\pi F_0[k-m]t) dt \\
&= \left. \frac{\exp(j2\pi F_0[k-m]t])}{j2\pi F_0[k-m]} \right|_{-T_0/2}^{T_0/2} \\
&= \frac{1}{F_0} \frac{\exp(j\pi[k-m]) - \exp(-j\pi[k-m])}{2j\pi[k-m]} \\
&= T_0 \frac{\sin(\pi[k-m])}{\pi[k-m]} \\
&= T_0 \delta[k-m] = \begin{cases} T_0, & k = m, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

CT complex exponential: property 6

- complex exponentials are eigenfunctions of CT-LTI systems



$$
\begin{aligned}
x(t) &= \exp(j2\pi F_0 t) \\
y(t) &= h(t) * x(t) = \int h(\tau)\exp(j2\pi F_0[t-\tau])d\tau \\
&= \underbrace{\int h(\tau)\exp(-j2\pi F_0 \tau)d\tau}_{H(F_0)} \cdot \underbrace{\exp(j2\pi F_0 t)}_{x(t)} = H(F_0)x(t)
\end{aligned}
$$

- output has same frequency as the input
- only amplitude and phase have changed: $H(F_0) = |H(F_0)|\exp(j\angle H(F_0))$

DT complex exponentials

- can be obtained by periodically sampling CT complex exponentials
- let $F_s = \frac{1}{T_s}$ be the sample rate/sample frequency
- sample times $t = nT_s$

$$x[n] = x(t)|_{t=nT_s} = \exp(j2\pi F_0 T_s n) = \exp(j2\pi f_0 n)$$

$$f_0 = F_0 T_s = \frac{F_0}{F_s} \qquad \text{normalized (cyclic) frequency [cycles/sample]}$$

$$\omega_0 = 2\pi f_0 = 2\pi \frac{\Omega_0}{F_s} = 2\pi \Omega_0 T_s \qquad \text{normalized frequency [rads/sample]}$$

CT versus DT complex exponentials

- CT complex exponential: $x(t) = \exp(j2\pi F_0 t)$
  - continuous frequency variable
  - continuous time variable

- DT complex exponential: $x[n] = \exp(j2\pi f_0 n)$
  - continuous frequency variable
  - discrete time variable

- the behavior is quite different due to discreteness of time

DT complex exponential: property 1

$$x[n] = \exp(j2\pi f_0 n)$$

- periodic only for rational frequencies $f_0 = \frac{k}{N}$
- period N

$$\begin{aligned} x[n+N] &= \exp(j2\pi f_0 (n+N)) \\ &= \exp(j2\pi f_0 n)\exp(j2\pi f_0 N) = x[n]\exp(j2\pi f_0 N) \end{aligned}$$

- for $x[n]$ to be periodic with period N, we must have

$$\exp(j2\pi f_0 N) = 1 \quad \Rightarrow \quad f_0 N = k \quad \text{(integer)}$$

- $f = k/N$ where k and N are relatively prime, i.e. $\gcd(k, N) = 1$

$$f_0 = \frac{F_0}{F_s} = \frac{T_s}{T_0} = \frac{k}{N}$$

$$kT_0 = NT_s \quad \text{(CT-CE complets } k \text{ periods in } N \text{ samples)}$$

$$kF_s = NF_0 \quad \text{(}N\text{th harmonic of CT-CE is a multiple of the sample rate)}$$

$$k/T_s = N/T_0 \quad \text{(num. samples/period multiple of num. samples/sec.)}$$

DT complex exponential: property 2

- non-uniqueness of DT-CE

- DT-CE are periodic in the frequency variable

$$\exp(j2\pi[f_0 + k]n) = \exp(j2\pi f_0 n)\underbrace{\exp(j2\pi kn)}_{1} = \exp(j2\pi f_0 n)$$

- f_0 and $f_0 + k$ are the same frequency (for any integer k)

- ω_0 and $\omega_0 + 2\pi k$ are the same frequency

- $\exp(j2\pi f_0 n)$ is periodic in f_0 with period 1

- $\exp(j\omega_0 n)$ is periodic in ω_0 with period 2π

- let $x_1[n] = \exp(j2\pi f_1 n)$ and $x_2[n] = \exp(j2\pi f_2 n)$, if $f_2 = f_1 + k$ then

$$x_1[n] = x_2[n] \quad \text{for all } n$$

- distinct frequencies lie in a fundamental interval (fundamental range)

$$
\begin{aligned}
f &\in [0, 1) \quad \text{or} \quad [-\tfrac{1}{2}, \tfrac{1}{2}) \\
\omega &\in [0, 2\pi) \quad \text{or} \quad [-\pi, \pi)
\end{aligned}
$$

CT complex exponential: property 3

$$x[n] = \exp(j2\pi f_0 n)$$

- the rate of oscillation does not increase indefinitely
- assuming fundamental range $[0, 1)$, rate of oscillation increases from 0 to $\frac{1}{2}$, and then it decreases from $\frac{1}{2}$ to 1
- "low" frequencies near 0 and ± 1 and ± 2 and ± 3 and ...
- "high" frequencies near $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$ and $\pm\frac{5}{2}$ and ...
  (half way in between the low frequencies)

CT complex exponential: property 4

- a time shift is equivalent to a phase shift

$$x[n - n_0] = \exp(j2\pi f_0[n - n_0]) = \exp(j[2\pi f_0 n - \varphi]), \quad \varphi = 2\pi f_0 n_0$$

- not all phase shifts correspond to integer sample shifts

CT complex exponential: property 5

- finite number of harmonically related and orthogonal complex exponentials
- let $s_k[n] = \exp(j2\pi(k/N)n)$ be a periodic DT-CE
- (k/N) and $(k/N) + 1 = (k+N)/N$ give the same sequence
- there are only N harmonically related DT-CE sequences, $s_k[n], k = 0, 1, \cdots, N-1$
- they are orthogonal

$$\sum_{n=0}^{N-1} s_k[n] s_m^*[n] = \sum_{n=0}^{N-1} \exp\left(\frac{j2\pi[k-m]n}{N}\right) = \frac{1 - \exp(j2\pi[k-m])}{1 - \exp\left(\frac{j2\pi[k-m]}{N}\right)}$$

$$= \frac{\exp(j\pi[k-m]) - \exp(-j\pi[k-m])}{\exp\left(\frac{j\pi[k-m]}{N}\right) - \exp\left(\frac{-j\pi[k-m]}{N}\right)} \exp\left(\frac{-j\pi[k-m][N-1]}{N}\right)$$

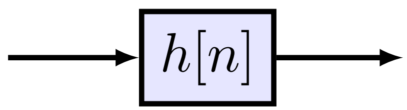
$$= \frac{\sin(\pi[k-m])}{\sin(\pi[k-m]/N)} \exp\left(\frac{-j\pi[k-m][N-1]}{N}\right) = N\delta[m-k]$$

- here is a useful identity that is easily verified by multiplication

$$\begin{aligned}
\exp(jA) - \exp(jB) &= \left[\exp\left(j\frac{A+B}{2}\right) - \exp\left(-j\frac{A+B}{2}\right)\right]\exp\left(j\frac{A-B}{2}\right) \\
&= 2j\sin\left(\frac{A+B}{2}\right)\exp\left(j\frac{A-B}{2}\right)
\end{aligned}$$

DT complex exponential: property 6

- complex exponentials are eigenfunctions of DT-LTI systems

$x[n] = \exp(j2\pi f_0 n)$  $y[n] = H(f_0)\exp(j2\pi f_0 n) = H(f_0)x[n]$

$$\begin{aligned} x[n] &= \exp(j2\pi f_0 n) \\ y[n] &= h[n] * x[n] = \sum h[k]\exp(j2\pi f_0[n-k]) \\ &= \underbrace{\sum h[k]\exp(-j2\pi f_0 k)}_{H(f_0)} \cdot \underbrace{\exp(j2\pi f_0 n)}_{x[n]} = H(f_0)x[n] \end{aligned}$$

- output has same frequency as the input

- only amplitude and phase have changed: $H(f_0) = |H(f_0)|\exp(j\angle H(f_0))$

frequency aliases of f_0'

- fundamental range: $[0, 1)$
  - take the fractional part: $f_0 = f_0' - \lfloor f_0' \rfloor$
  - ex: $f_0' = 36.1$ aliases to $f_0 = 0.1$ which is a low frequency
  - ex: $f_0' = 36.9$ aliases to $f_0 = 0.9$ which is a low frequency

- fundamental range: $[-0.5, 0.5)$
  - take the fractional part: $f_0 = f_0' - \left\lfloor f_0' + \frac{1}{2} \right\rfloor$
  - ex: $f_0' = 36.1$ aliases to $f_0 = 0.1$ which is a low frequency
  - ex: $f_0' = 36.9$ aliases to $f_0 = -0.1$ which is a low frequency

- the basic idea is to keep subtracting 1 from f_0' until the answer falls into the fundamental range

frequency aliases when sampling

- Q: a sinusoid with frequency F_0' is sampled at a rate F_s, what is the perceived frequency?

$$F_0' \xrightarrow{\text{sampled at } F_s} f_0' = \frac{F_0'}{F_s}$$

$$f_0' = \frac{F_0'}{F_s} \;\Big\downarrow\; \text{alias}$$

$$F_0 = f_0 F_s \xleftarrow{\text{perceived as}} f_0 = f_0' - \lfloor f_0' + 0.5 \rfloor$$

$$f_0 = \frac{F_0'}{F_s} - \left\lfloor \frac{F_0'}{F_s} + \frac{1}{2} \right\rfloor \qquad\qquad F_0 = F_0' - F_s \left\lfloor \frac{F_0'}{F_s} + \frac{1}{2} \right\rfloor$$

DT complex exponential: property 7

- small changes in frequency can lead to large changes in the period
- ex: $\cos(2\pi(12/36)n) = \cos(2\pi n/3)$ has frequency $12/36 = 1/3$ and fundamental period $N = 3$
- ex: $\cos(2\pi(13/36)n)$ has frequency $13/36$ and fundamental period $N = 36$
- Q: what is going on here?
- A1: the "envelope" (CT-sinusoid) of the signal changes only slightly in these examples
- A2: whether the sampled signal repeats (periodic) or not depends on the relation between the sample rate and the frequency

continuous-time Fourier series

- for periodic signals $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T} \qquad X_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-j2\pi kt/T}dt$$

- consider the special case when the period is $T = 1$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt} \qquad X_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} x(t)e^{-j2\pi kt}dt$$

Dirichlet (sufficient) conditions

1. $\int_T |x(t)|dt < \infty \quad \Rightarrow \quad |X_k| < \infty$

2. $x(t)$ has a finite number of maxima and minima in any period

3. $x(t)$ has a finite number of discontinuities in any period

- if these conditions are satisfied, then Fourier sum converges to $x(t)$ at all points where $x(t)$ is continuous and converges to the average value of the right-hand and left-hand limits at points where $x(t)$ is discontinuous

- necessary conditions are not known

- every physically real periodic signal satisfies 1, 2, 3

- every physically real periodic signal has CTFS

- sometimes CTFS can be calculated by hand

- most of the time CTFS is evaluated numerically

continuous-time Fourier transform

- for aperiodic signals $x(t)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \qquad X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}dF \qquad X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$$

Dirichlet (sufficient) conditions

1. $\int |x(t)|dt < \infty \quad \Rightarrow \quad |X(F)| < \infty$

2. $x(t)$ has a finite number of maxima and minima

3. $x(t)$ has a finite number of discontinuities

- if these conditions are satisfied, then Fourier integral converges to $x(t)$ at all points where $x(t)$ is continuous and converges to the average value of the right-hand and left-hand limits at points where $x(t)$ is discontinuous

- necessary conditions are not known

- every physically real signal satisfies 1, 2, 3

- every physically real signal has CTFT

- sometimes CTFT can be calculated by hand

- most of the time CTFT is evaluated numerically

CTFT pairs

$x(t)$	$X(\Omega)$	$X(F)$
$\delta(t)$	1	???
1	$2\pi\delta(\Omega)$	
$\delta(t-t_0)$	$e^{-j\Omega t_0}$	
$e^{j\Omega_0 t}$	$2\pi\delta(\Omega-\Omega_0)$	
$\cos(\Omega_0 t)$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$	
$\sin(\Omega_0 t)$	$j\pi[-\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$	
$u(t)$	$\pi\delta(\Omega)+\frac{1}{j\Omega}$	
$\delta(t)-\frac{1}{jt}$	$2\pi u(\Omega)$	
$\mathsf{sgn}(t)$	$\frac{2}{j\Omega}$	
$\begin{cases}1, & \lvert t\rvert \le t_0\\ 0, & \lvert t\rvert > t_0\end{cases}$	$\frac{2\sin(\Omega t_0)}{\Omega}$	
$\frac{\sin(\Omega_0 t)}{\pi t}$	$\begin{cases}1, & \lvert\Omega\rvert \le \Omega_0\\ 0, & \lvert\Omega\rvert > \Omega_0\end{cases}$	
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-\frac{2\pi}{T}k\right)$	
$e^{-\frac{t^2}{2\sigma^2}}$	$\sqrt{2\pi\sigma^2}e^{-\frac{\sigma^2\omega^2}{2}}$	

CTFT of periodic signals

- let $x(t)$ have finite energy and be aperiodic
- $y(t) = \sum_n x(t - nT_0)$ is periodic with period T_0
- write it as a convolution:

$$y(t) = x(t) * \sum_n \delta(t - nT_0) \tag{1}$$

- note that the CTFT of $\sum_n \delta(t - nT_0)$ is $F_0 \sum_k \delta(F - kF_0), \quad F_0 = 1/T_0$
- take CTFT of both sides of (1)

$$Y(F) = X(F) \cdot F_0 \sum_k \delta(F - kF_0) = F_0 \sum_k X(kF_0)\delta(F - kF_0)$$

- periodic in time-domain leads to weighted impulse train in the frequency-domain (and vice versa)
- periodic replication in the time-domain leads to samping in the frequency domain (and vice versa)

discrete-time Fourier transform

- for aperiodic sequences $x[n]$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \qquad X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi f n} df \qquad X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

- DTFT $X(f)$ is periodic with period 1

- DTFT $X(\omega)$ is periodic with period 2π

$$X(f+k) = \sum_n x[n] e^{-j2\pi(f+k)n} = \sum_n x[n] e^{-j2\pi f n} \underbrace{e^{-j2\pi k n}}_{1} = X(f)$$

- $X(f)$ is called the spectrum of $x[n]$; $|X(f)|$ is the magnitude spectrum; $\angle X(f)$ is the phase spectrum

DTFT - CTFS duality

- continuous-time Fourier series for the special case when the period is 1

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt} \qquad X_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} x(t) e^{-j2\pi kt} dt$$

(synthesis) (analysis)

- discrete-time Fourier transform has period equal to 1

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df \qquad X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

(synthesis) (analysis)

- $x[n] = X_k|_{k=-n}$

- $X(f) = x(t)|_{t=-f}$

DTFT convergence: three classes of signals

- absolutely summable signals

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- energy signals (square summable, finite energy, zero power)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- power signals (mean-square summable, infinte energy, finite power)

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty$$

DTFT convergence

- for the DTFT to exist, the sequence of partial sums

$$X_N(\omega) = \sum_{n=-N}^{N} x[n]e^{-j\omega n}$$

must converge to a finite limit for all ω as $N \to \infty$

- DTFT analysis formula

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \lim_{N\to\infty} \sum_{n=-N}^{N} x[n]e^{-j\omega n} = \lim_{N\to\infty} X_N(\omega)$$

DTFT convergence: absolutely summable signals

- suppose $x[n]$ is absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then
  - the DTFT $X(\omega)$ exists because

$$|X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| \, |e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  - the sequence of partial sums converges uniformly (point-wise) to $X(\omega)$

$$\lim_{N \to \infty} |X_N(\omega) - X(\omega)| = 0 \;\text{ for all }\; \omega$$

  - the DTFT $X(\omega)$ is a continuous function of ω
  - the DTFT $X(\omega)$ is infinitely differentiable, i.e. $\frac{d^k X(\omega)}{d\omega^k}$ exists for all $k \geq 1$

- for absolutely summable signals compute $X(\omega)$ by direct evaluation of the analysis equation

DTFT example: absolutely summable signals

- example: $x[n] = \delta[n] \quad \Rightarrow \quad X(\omega) = 1$
  - $x[n]$ is absolutely summable
  - is $X(\omega)$ continuous?
  - is $X(\omega)$ infintely differentiable?
  - is $X(\omega)$ periodic?

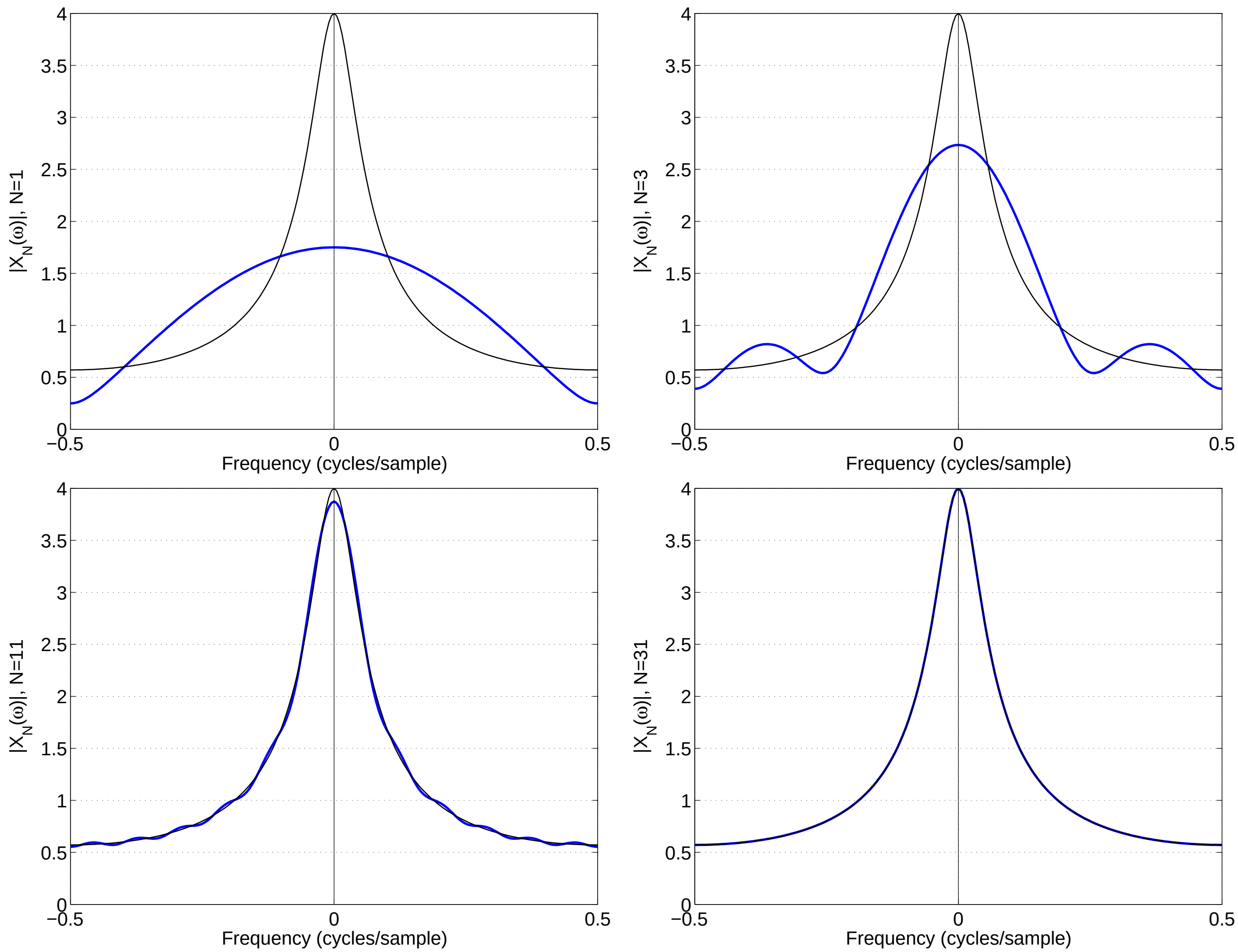
- example: $x[n]$ is any finite length signal

$$x(n) = \begin{cases} a_n, & N_0 \leq n \leq N_1, \\ 0, & \text{otherwise} \end{cases} = \sum_{k=N_0}^{N_1} a_n \delta(n-k) \quad \Rightarrow \quad X(\omega) = \sum_{n=N_0}^{N_1} a_n e^{-j\omega n}$$

  - $x[n]$ is absolutely summable
  - is $X(\omega)$ continuous?
  - is $X(\omega)$ infintely differentiable?
  - is $X(\omega)$ periodic?

DTFT example: absolutely summable signals

- example: $x[n] = a^n u[n]$ where $|a| < 1$, $X(\omega) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$
- point-wise convergence of partial sums $(N = 1, 3, 11, 31)$



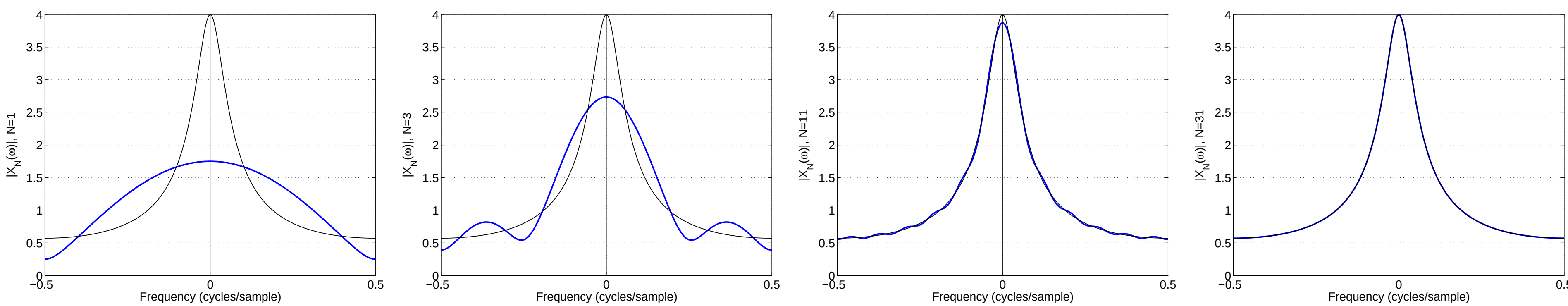
DTFT example: absolutely summable signals

- example: $x[n] = a^n u[n]$ where $|a| < 1$

$$X(\omega) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

  - $x[n]$ is absolutely summable
  - is $X(\omega)$ continuous?
  - is $X(\omega)$ infintely differentiable?
  - is $X(\omega)$ periodic?

- point-wise convergence of partial sums $(N = 1, 3, 11, 31)$



DTFT convergence: energy signals

- suppose $x[n]$ is an energy signal, i.e. $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$
- the sequence of partial sums converges in the mean-square sense to $X(\omega)$

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 \, d\omega = 0$$

- the interpretation is that the energy in the error tends to zero as $N \to \infty$
- for energy signals $X(\omega)$ has discontinuities
- at points of discontinuity ω, $X_N(\omega)$ converges to the average of the left-hand and right-hand limits
- for energy signals it is often very difficult to compute $X(\omega)$ using the analysis equation
- investigate DTFT pairs using the synthesis equation

DTFT example: energy signals

- example: $x[n] = \frac{\sin \omega_0 n}{\pi n}$
  - this is not absolutely summable
  - this is an energy signal
  - try to compute $X(\omega)$ using the analysis equation

$$X(\omega) = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_0 n}{\pi n} e^{-j\omega n} = \ ???$$

- compute the inverse-DTFT of the rectangular function

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

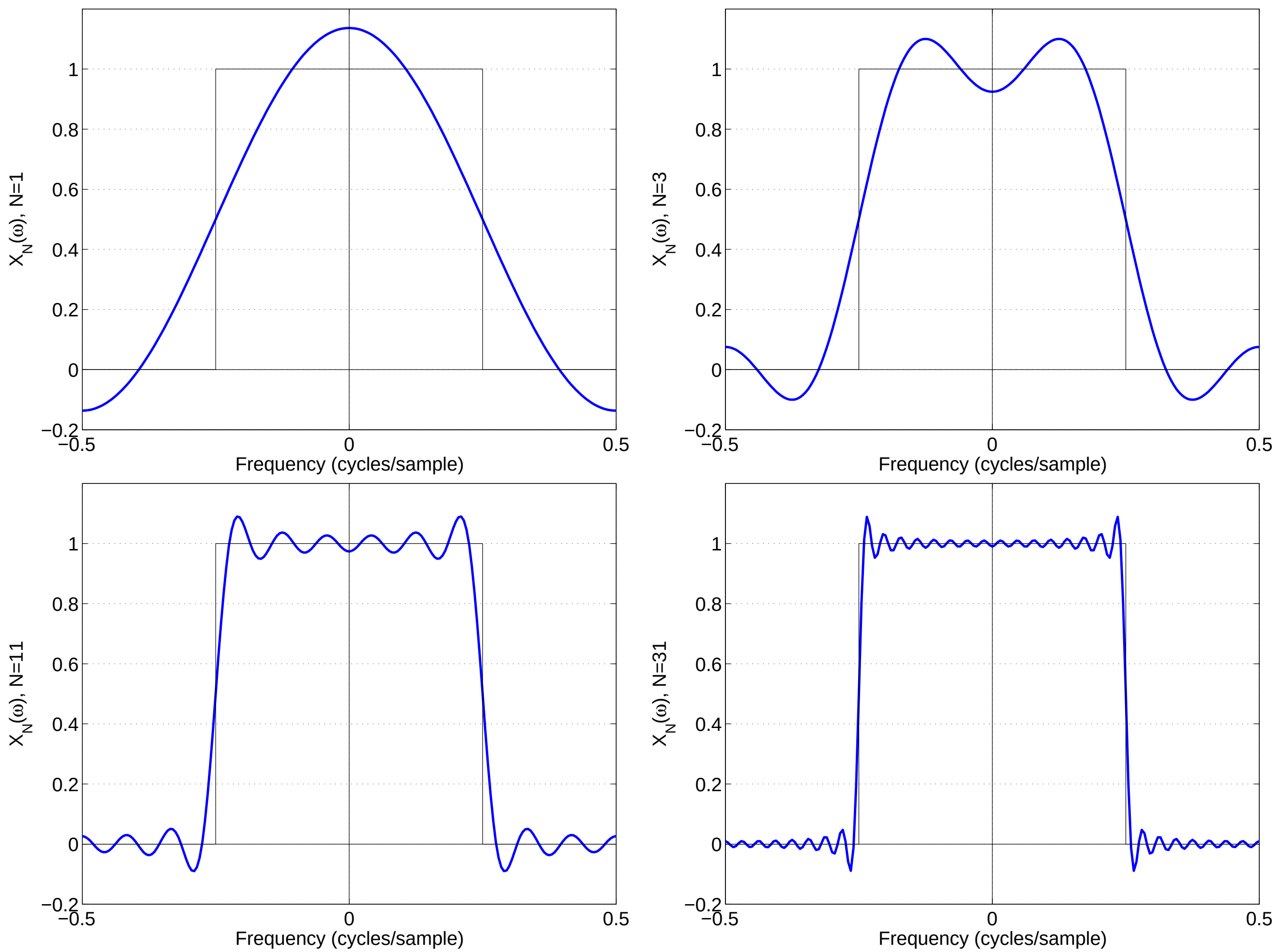
$$x[n] = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{j2\pi n} = \frac{\sin \omega_0 n}{\pi n} \quad \text{for all } n$$

- the inverse-DTFT (synthesis equation) was easy

DTFT example: energy signals

- example: $x[n] = \dfrac{\sin \omega_0 n}{\pi n} \quad \Rightarrow \quad X(\omega) = \begin{cases} 1, & |\omega| \le \omega_0 \le \pi \\ 0, & \text{otherwise} \end{cases}$

- mean-square convergence of partial sums $(N = 1, 3, 11, 31)$

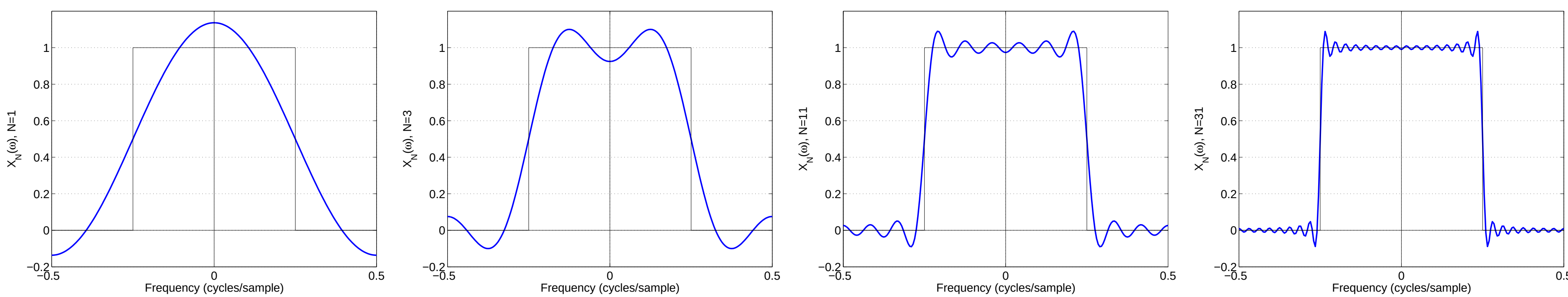


DTFT example: energy signals

- example:

$$x[n] = \frac{\sin \omega_0 n}{\pi n} \Rightarrow \quad X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

  - $x[n]$ is an energy signal
  - is $X(\omega)$ continuous?
  - is $X(\omega)$ infintely differentiable?
  - is $X(\omega)$ periodic?
  - at points ω of discontinuity, $X_N(\omega)$ converges to average of left-hand and right-hand limits
  - the oscillatory behavior of $X_N(\omega)$ near the discontinuity in $X(\omega)$ is referred to as the Gibbs phenomenon



DTFT convergence: power signals

- suppose $x[n]$ is an power signal, i.e. $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty$
- power signals do not have a DTFT
- by allowing Dirac impulse, $\delta(\omega)$, we can define DTFTs for some power signals: periodic signals, unit step
- for power signals $X(\omega)$ can not be computed by direct application of the analysis equation
- investigate DTFT pairs using the synthesis equation

DTFT example: power signal

- compute the inverse-DTFT of

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) e^{j\omega n} d\omega \\ &= \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \delta(\omega - 2\pi k) e^{j\omega n} d\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega \\ &= e^{j0n} = 1. \end{aligned}$$

- we have derived the DTFT pair: $x[n] = 1 \quad \Leftrightarrow \quad X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

DTFT example: power signal

- we can also derive the DTFT pair

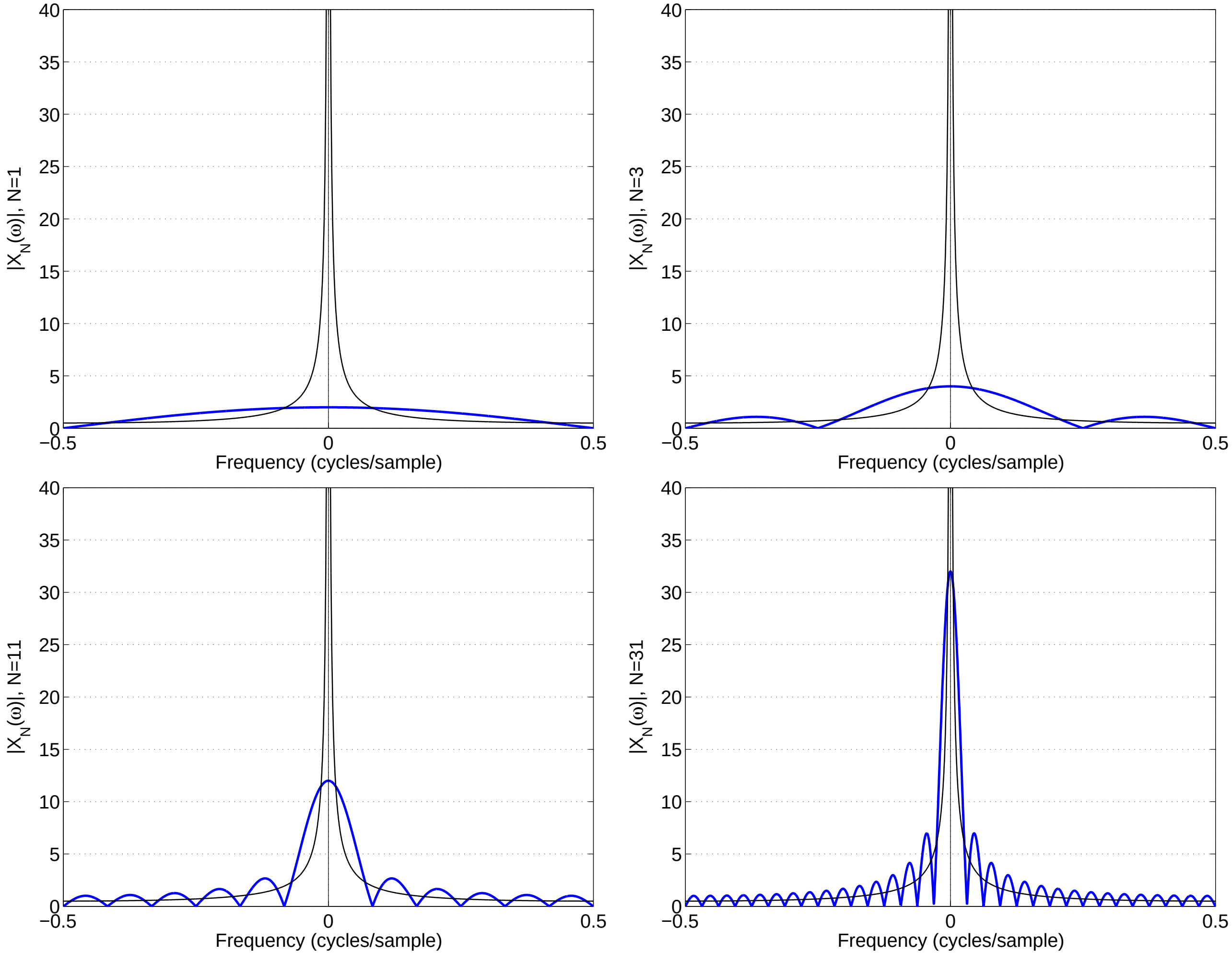
$$x[n] = e^{j\omega_0 n} \quad \Leftrightarrow \quad X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$$

  - $x[n]$ is a power signal
  - is $X(\omega)$ continuous?
  - is $X(\omega)$ infintely differentiable?
  - does $X(\omega)$ have finite discontinuitites?
  - $X(\omega)$ has Dirac delta functions
  - is $X(\omega)$ periodic?

DTFT example: power signal

- the unit step function has the DTFT

$$u[n] \quad \Leftrightarrow \quad U(\omega) = \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$



DTFT convergence

$x(n)$	$X(\omega)$	Notes
Absolutely summable signals $\sum_{n=-\infty}^{\infty} \lvert x(n) \rvert < \infty$	Uniform convergence Continuous $X(\omega)$ Differentiable $X(\omega)$	Compute DTFT directly
Energy signals $\sum_{n=-\infty}^{\infty} \lvert x(n) \rvert^2 < \infty$	Mean-square convergence Jump discontinuities	Verify DTFT by IDTFT
Power signals $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x(n) \rvert^2 < \infty$	May not converge in any sense Converges at some frequencies for periodic signals Converges at some frequencies for unit step May include Dirac impulses	Verify DTFT by IDTFT

discrete-time Fourier series (DTFS)

- suppose $x[n]$ is periodic with period N

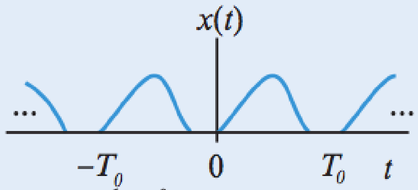
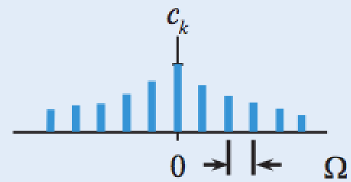
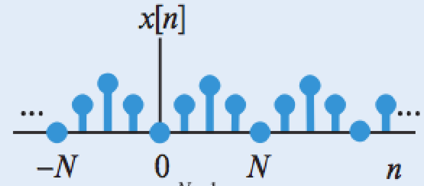
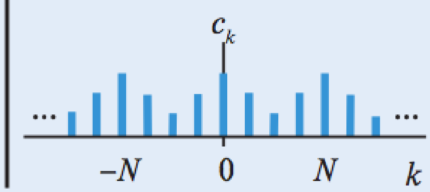
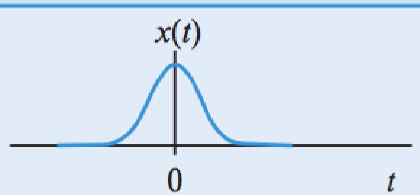
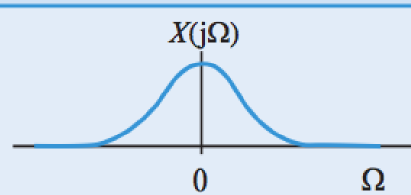
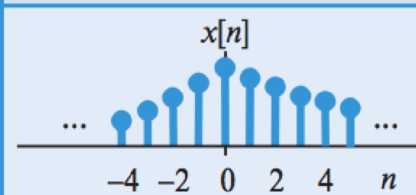
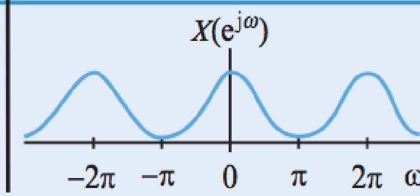
$$x[n] = \sum_{k=0}^{N-1} X_k e^{\frac{j2\pi kn}{N}} \qquad\qquad X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}$$

- we'll have a lot more to say about the DTFS later when we talk about the discrete Fourier transform (DFT)

- always converges (finite sum of finite numbers)

summary

Table 4.1 Summary of Fourier representation of signals.

		Continuous-time signals: Time-domain	Continuous-time signals: Frequency-domain	Discrete-time signals: Time-domain	Discrete-time signals: Frequency-domain
Periodic signals	Fourier series	 $c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\Omega_0 t}dt$	 $\Omega_0 = \frac{2\pi}{T_0}$ $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$ CTFS →, ← ICTFS	 $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$	 $x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$ DTFS →, ← IDTFS
		Continuous and periodic	Discrete and aperiodic	Discrete and periodic	Discrete and periodic
Aperiodic signals	Fourier transforms	 $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	 $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}d\Omega$ CTFT →, ← ICTFT	 $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$	 $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ DTFT →, ← IDTFT
		Continuous and aperiodic	Continous and aperiodic	Discrete and aperiodic	Continous and periodic

- discreteness in one domain leads to periodicity in the other domain
- periodicity in one domain leads to discreteness in the other domain

Parseval relations

CTFS: $$\frac{1}{T_0}\int_{T_0}|x(t)|^2dt = \sum_{k=-\infty}^{\infty}|X_k|^2$$

CTFT: $$\int_{-\infty}^{\infty}|x(t)|^2dt = \int_{-\infty}^{\infty}|X(F)|^2dF$$

DTFT: $$\sum_{n=-\infty}^{\infty}|x[n]|^2dt = \int_{-1/2}^{1/2}|X(f)|^2df$$

DTFS: $$\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \sum_{k=0}^{N-1}|X_k|^2$$

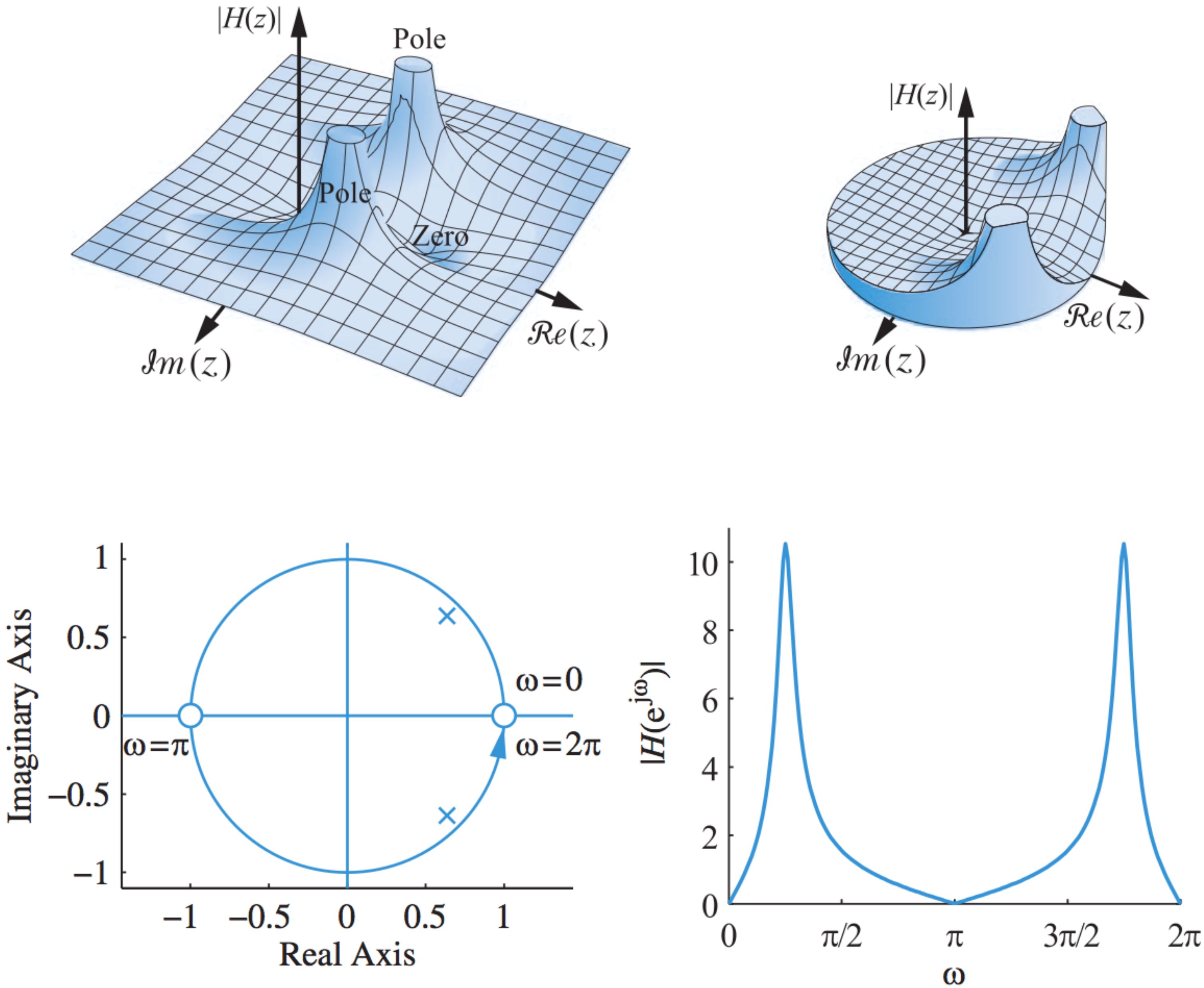
z-transform and DTFT

- compare the z-transform and the DTFT

$$X(z) = \sum_{n=\infty}^{\infty} x[n] z^{-n}$$

$$X(\omega) = \sum_{n=\infty}^{\infty} x[n] e^{-j\omega n}$$

- we see that $X(\omega) = X(z)|_{z=e^{j\omega}}$
- this requires that the region of convergence of $X(z)$ include the unit circle in the z-plane
- only the DTFTs of absolutely summable signals can be generated in this way
- this does not apply to energy signals or power signals



|H(z)|
Pole
Pole
Zero
ℛe(z)
ℐm(z)
|H(z)|
ℛe(z)
ℐm(z)
Imaginary Axis
1
0.5
0
−0.5
−1
ω=0
ω=π
ω=2π
−1
−0.5
0
0.5
1
Real Axis
|H(e^{jω})|
10
8
6
4
2
0
0
π/2
π
3π/2
2π
ω

Table 4.3 Symmetry properties of the DTFT.

Sequence $x[n]$	Transform $X(e^{j\omega})$
	Complex signals
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_R[n]$	$X_e(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$jx_I[n]$	$X_o(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_e[n] \triangleq \frac{1}{2}(x[n] + x^*[-n])$	$X_R\left(e^{j\omega}\right)$
$x_o[n] \triangleq \frac{1}{2}(x[n] - x^*[-n])$	$jX_I\left(e^{j\omega}\right)$
	Real signals
Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_R\left(e^{j\omega}\right) = X_R\left(e^{-j\omega}\right)$ $X_I\left(e^{j\omega}\right) = -X_I\left(e^{-j\omega}\right)$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
$x_e[n] = \frac{1}{2}(x[n] + x[-n])$ Even part of $x[n]$	$X_R\left(e^{j\omega}\right)$ real part of $X(e^{j\omega})$ (even)
$x_o[n] = \frac{1}{2}(x[n] - x[-n])$ Odd part of $x[n]$	$jX_I\left(e^{j\omega}\right)$ imaginary part of $X(e^{j\omega})$ (odd)

DTFT is a 4-way transform

Table 4.2 Special cases of the DTFT for real signals.

Signal	Fourier transform
Real and even	real and even
Real and odd	imaginary and odd
Imaginary and even	imaginary and even
Imaginary and odd	real and odd

$$x(n) = x^{re}(n) + jx^{ie}(n) + x^{ro}(n) + jx^{io}(n)$$

$$X(\omega) = X^{re}(\omega) + jX^{ie}(\omega) + X^{ro}(\omega) + jX^{io}(\omega)$$