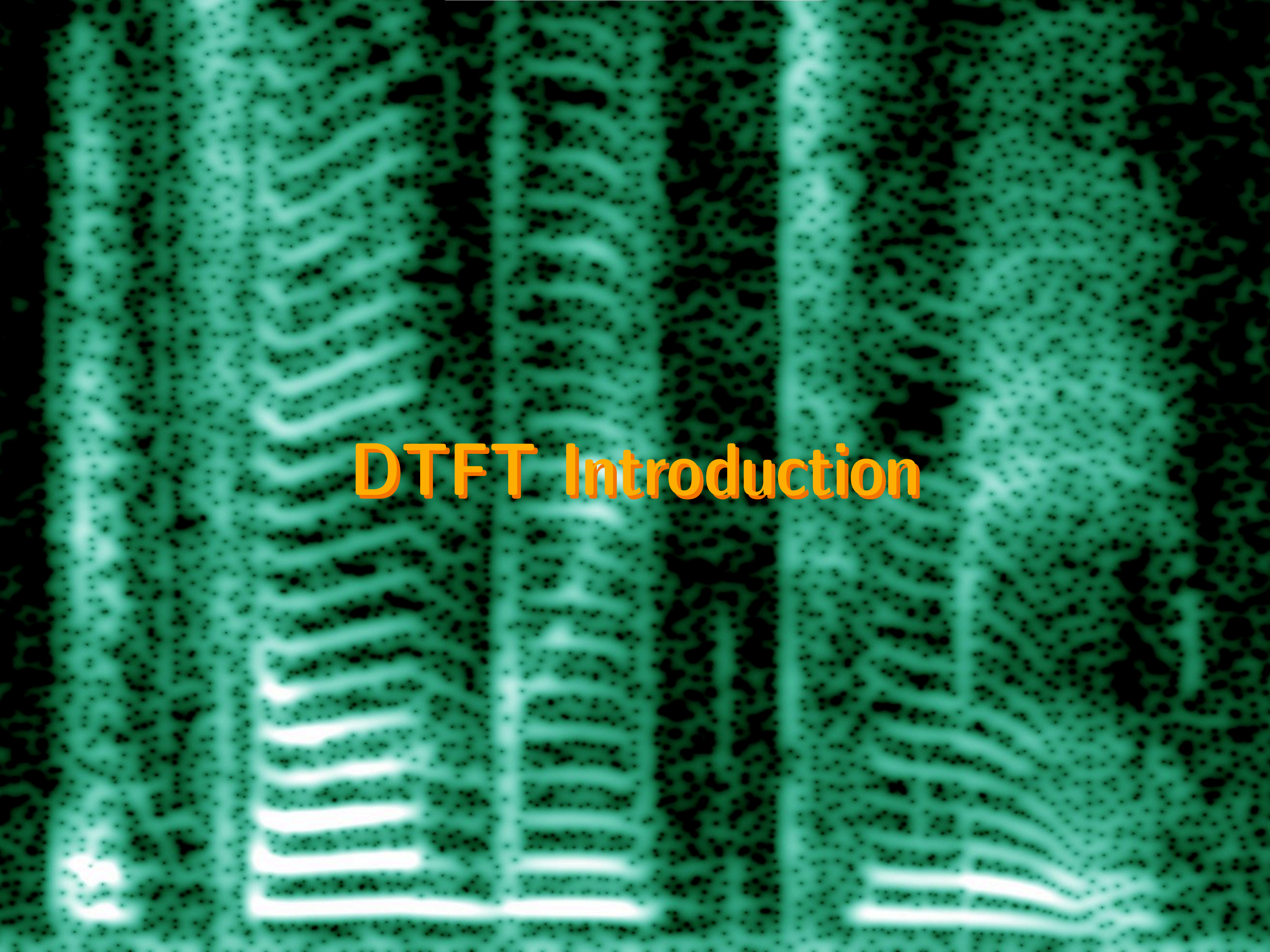
Discrete-Time Fourier Transform

ECE 3640 Discrete-Time Signals and Systems

Jake Gunther

DTFT Introduction



Family of Fourier Transforms

	frequency: continuous time: aperiodic	frequency: discrete time: periodic
frequency: aperiodic time: continuous	$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$ $X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$ CTFT	$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}}$ $X[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j\frac{2\pi kt}{T}} dt$ CTFS
frequency: periodic time: discrete	DTFT $x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn}$ $X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$	DTFS/DFT/FFT $x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$ $X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$

Family of Fourier Transforms

	frequency: continuous time: aperiodic	frequency: discrete time: periodic
frequency: aperiodic time: continuous	CTFT	CTFS
frequency: periodic time: discrete	DTFT	DTFS/DFT/FFT

CTFT

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}}$$

$$X[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j\frac{2\pi kt}{T}} dt$$

DTFT

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

DTFS/DFT/FFT

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

CTFS ↔ DTFT: $T = 1, t = -f, k = n$

DTFT is Periodic with Period 1

DFTF synthesis formula $$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn}$$

DFTF analysis formula $$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

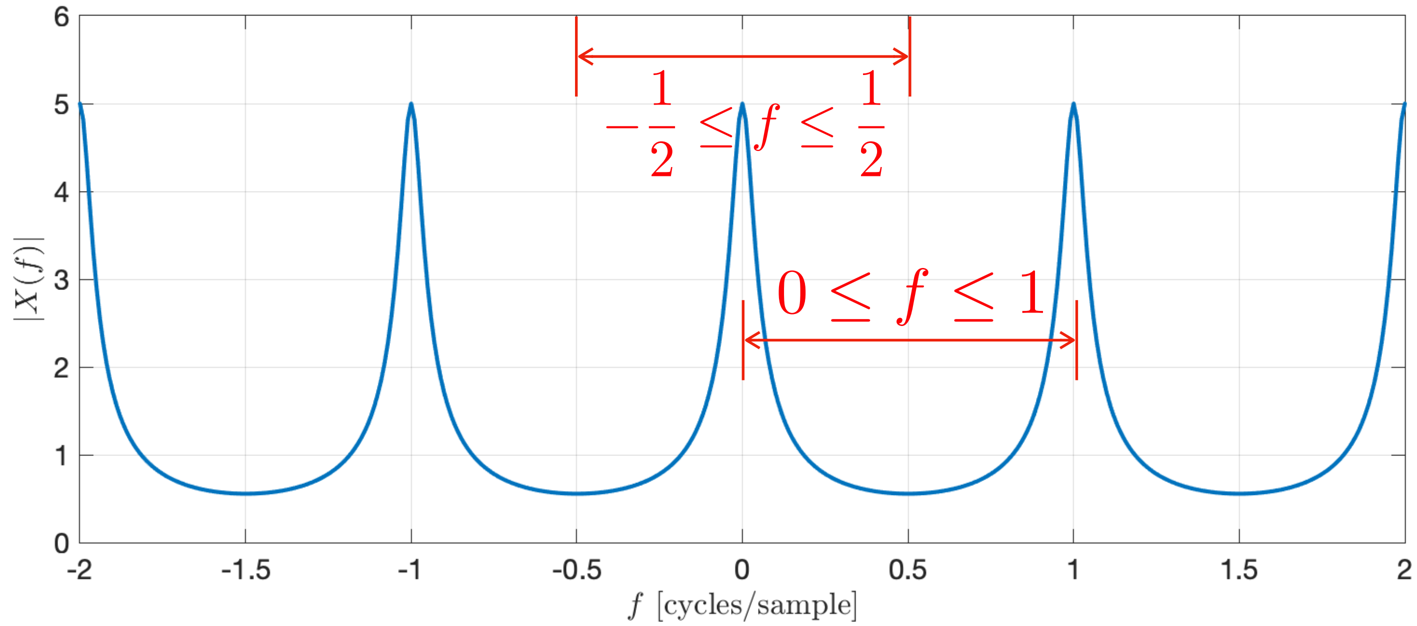
The DTFT is periodic with period 1.

$$\begin{aligned} X(f+1) &= \sum_n x[n] e^{-j2\pi(f+1)n} \\ &= \sum_n x[n] e^{-j2\pi fn} e^{-j2\pi n} \\ &= \sum_n x[n] e^{-j2\pi fn} = X(f) \end{aligned}$$

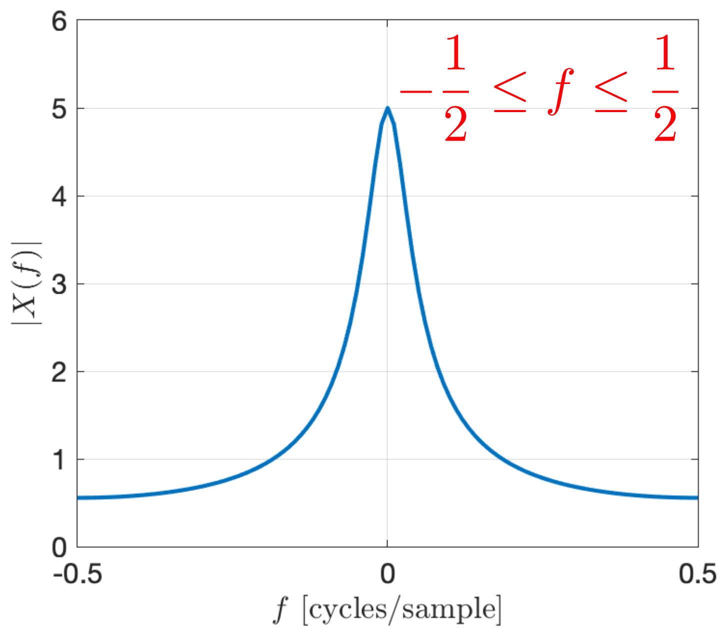
In general, we have $X(f+k) = X(f)$ for any integer k.

DTFT is Periodic with Period 1

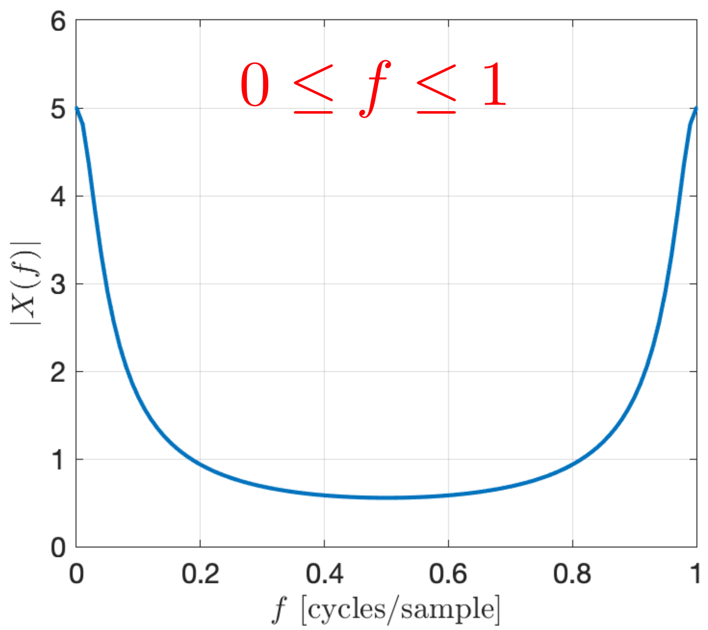
The DTFT is periodic with period 1.



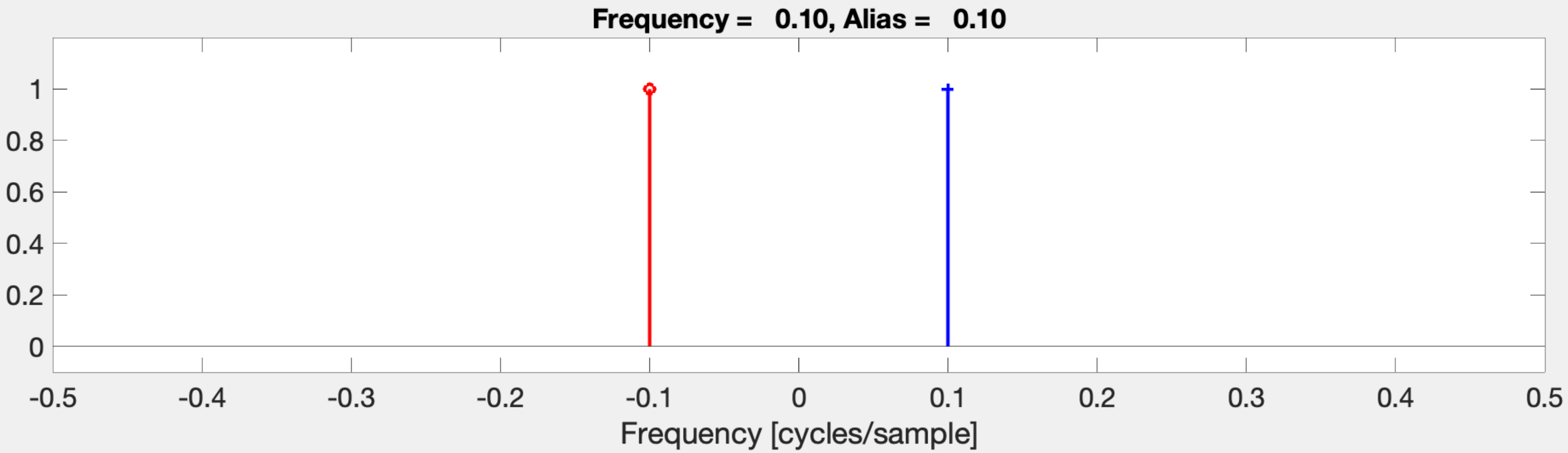
$$x[n] = 0.8^n u[n] \quad \longleftrightarrow \quad X(f) = \frac{1}{1 - 0.8e^{-j2\pi f}}$$



Due to periodicity, we usually only plot one period of $X(f)$. But don't forget that it's really a periodic function.

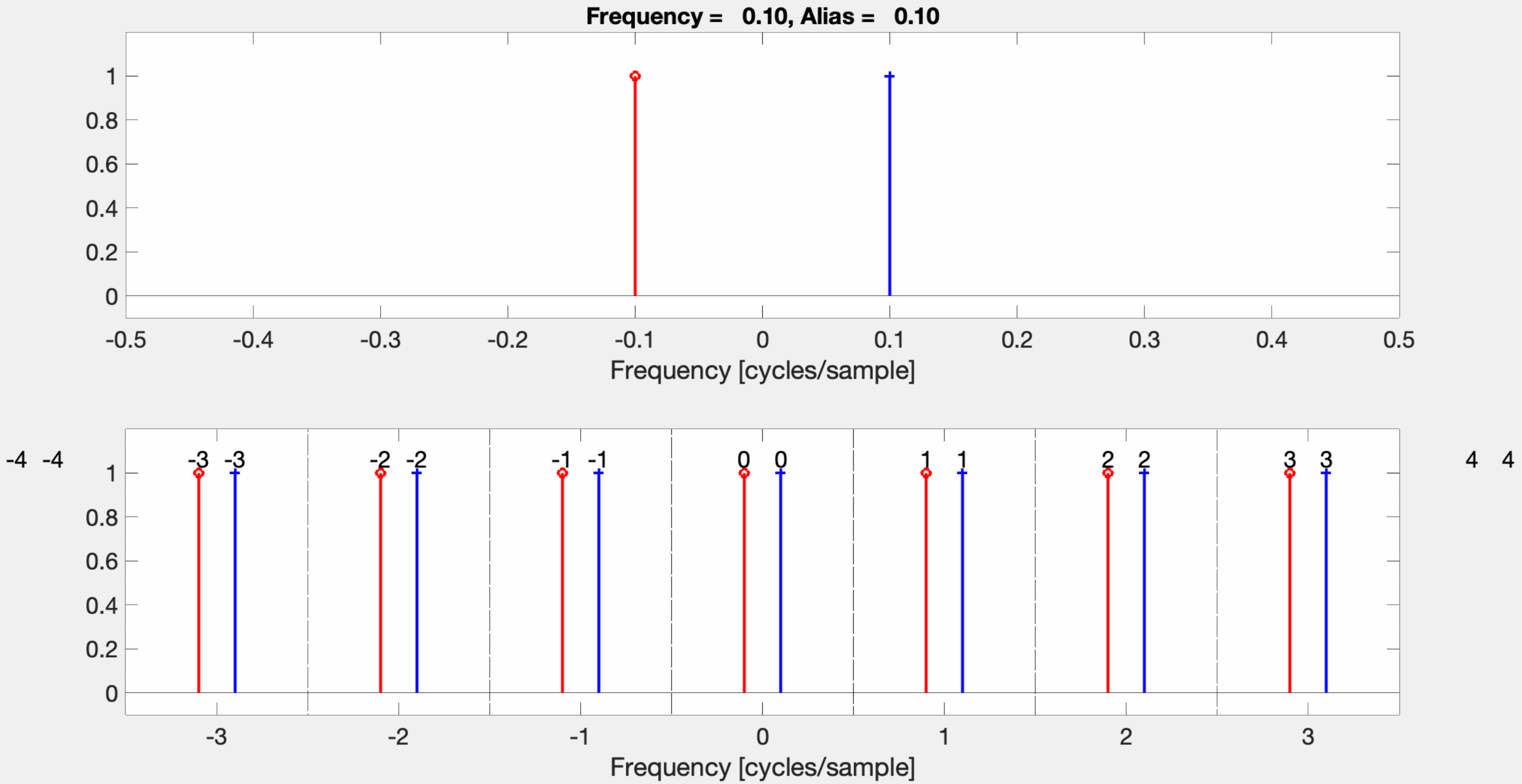


Periodicity of DTFT Explains Frequency Aliasing



$$\cos(2\pi f_0 n) \iff \frac{1}{2}\sum_k \delta(f - f_0 - k) + \frac{1}{2}\sum_k \delta(f + f_0 - k)$$

Periodicity of DTFT Explains Frequency Aliasing



$$\cos(2\pi f_0 n) \iff \frac{1}{2}\sum_k \delta(f - f_0 - k) + \frac{1}{2}\sum_k \delta(f + f_0 - k)$$

DTFT Frequency Variable

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df$$

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

f [cycles/sample]

period 1

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

ω [radians/sample]

period 2π

Derive DTFT by Convolution

Consider an LTI system with impulse response $h[n]$.
Compute the output when the input is $x[n] = e^{j2\pi fn}$.

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j2\pi f(n-k)} \\
&= \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j2\pi fk)}\right) e^{j2\pi fn} \\
&= H(f)e^{j2\pi fn} \\
H(f) &= \sum_{k=-\infty}^{\infty} h[k]e^{-j2\pi fk}
\end{aligned}
$$

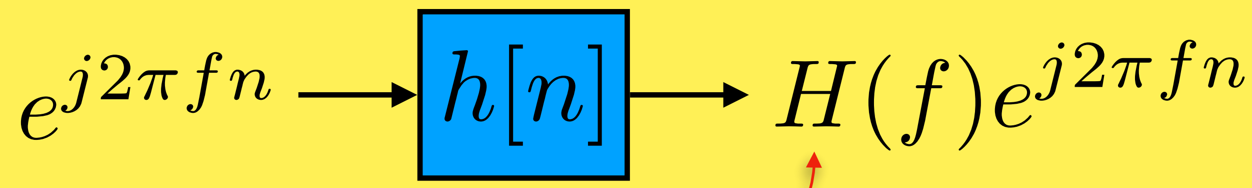
Derive DTFT by Convolution

Consider an LTI system with impulse response $h[n]$.
Compute the output when the input is $x[n] = e^{j2\pi fn}$.

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j2\pi f(n-k)} \\
&= \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j2\pi fk} \right) e^{j2\pi fn} \\
&= H(f)e^{j2\pi fn}
\end{aligned}
$$

$$
H(f) = \sum_{k=-\infty}^{\infty} h[k]e^{-j2\pi fk}
$$

Eigenfunction property of LTI systems



$H(f)$ gives the response of the system when the input is an everlasting complex exponential at frequency f.

LTI system only modifies the magnitude and phase of an everlasting complex exponential input.

$H(f)$ is the DTFT of the impulse response $h[n]$.
$H(f)$ is called the frequency response.

LTI System Frequency Response

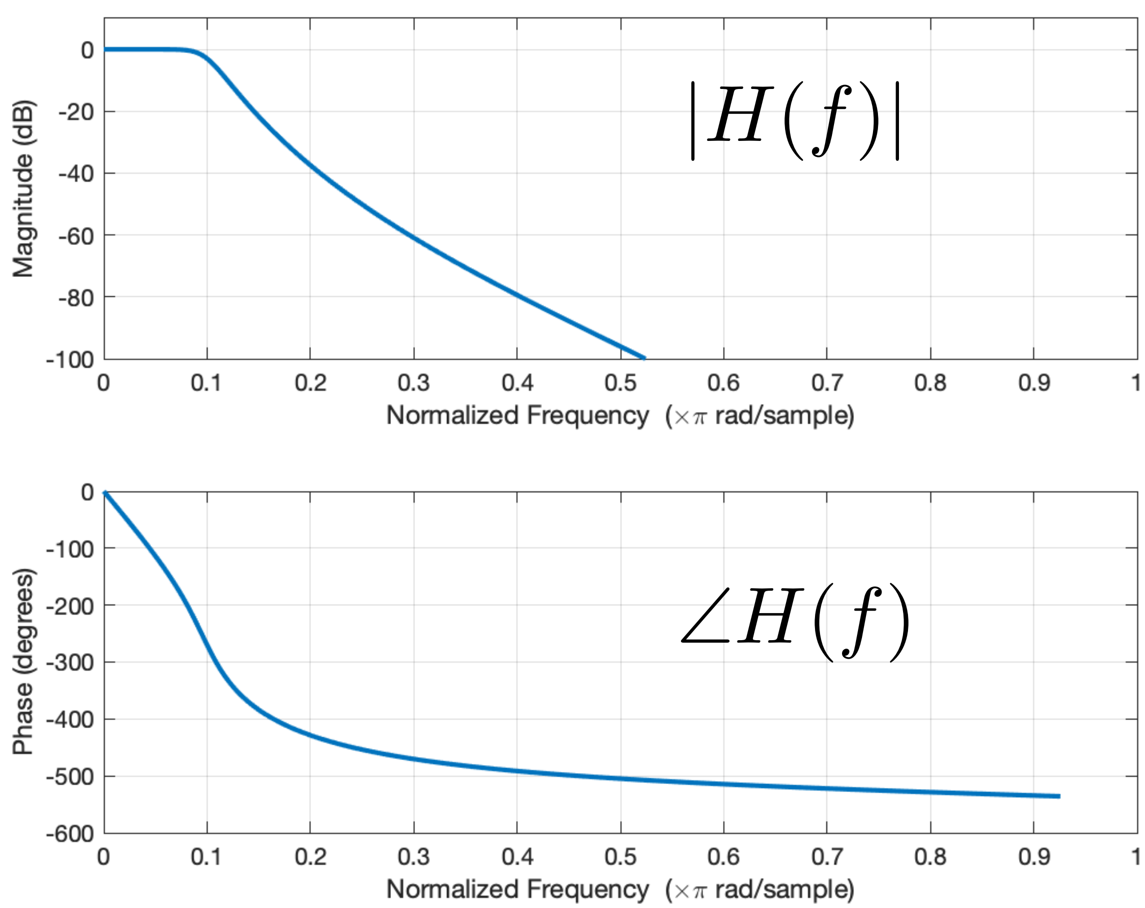
$$e^{j2\pi fn} \longrightarrow \boxed{h[n]} \longrightarrow H(f)e^{j2\pi fn}$$

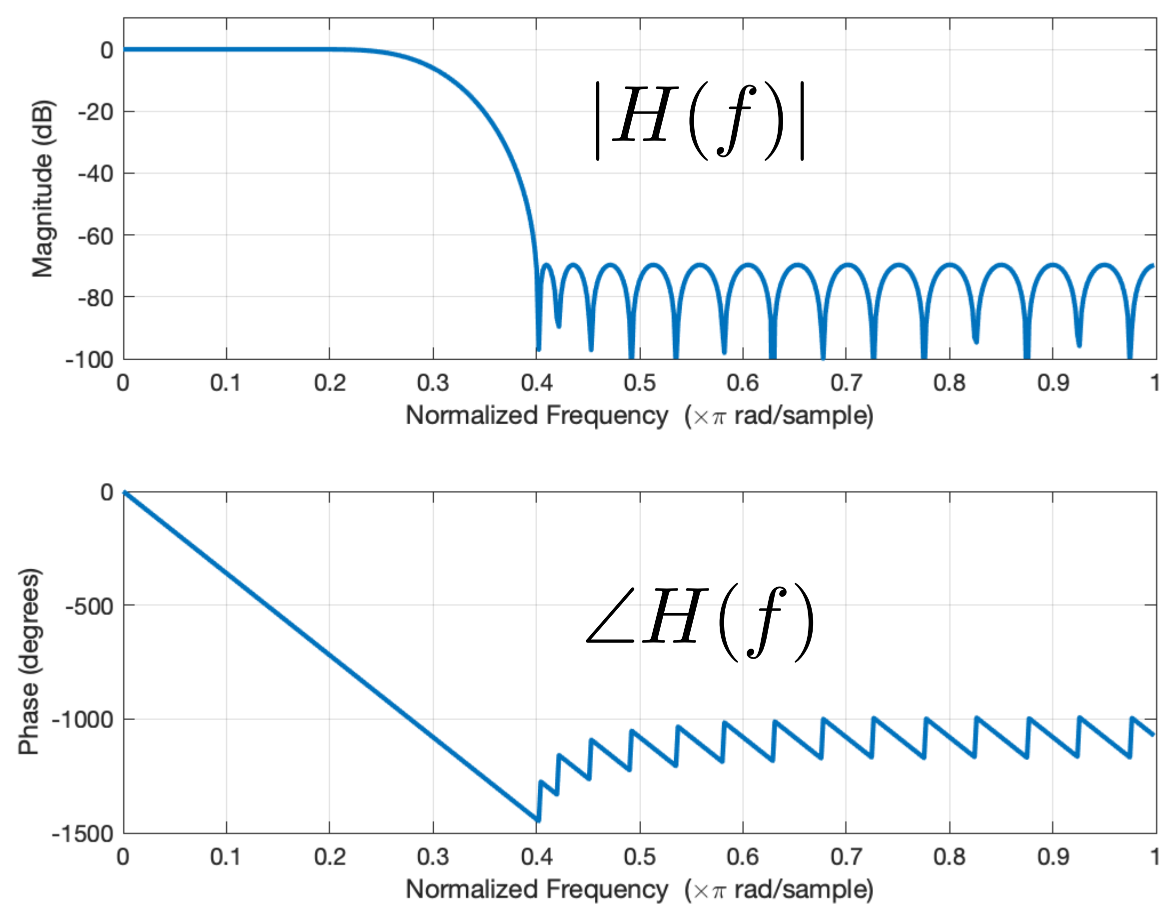
$$H(f) = |H(f)|e^{j\angle H(f)}$$

$|H(f)|$ = magnitude response

$\angle H(f)$ = phase response

$H(f)$ describes how everlasting complex exponentials are modified as a function of their frequencies.





LTI System Responses

Frequency Response

$$H(f) = |H(f)|e^{j\angle H(f)}$$

Impulse Response

$$h[n]$$

The impulse response $h[n]$ describes how the system modifies an impulse $\delta[n]$. For a general input $x[n]$, decompose into linear combinations of train of delayed deltas

$$x[n] = \sum_k x[k]\delta[n-k]$$

and by LTI properties, the output is

$$y[n] = \sum_k x[k]h[n-k] = x[n] * h[n].$$

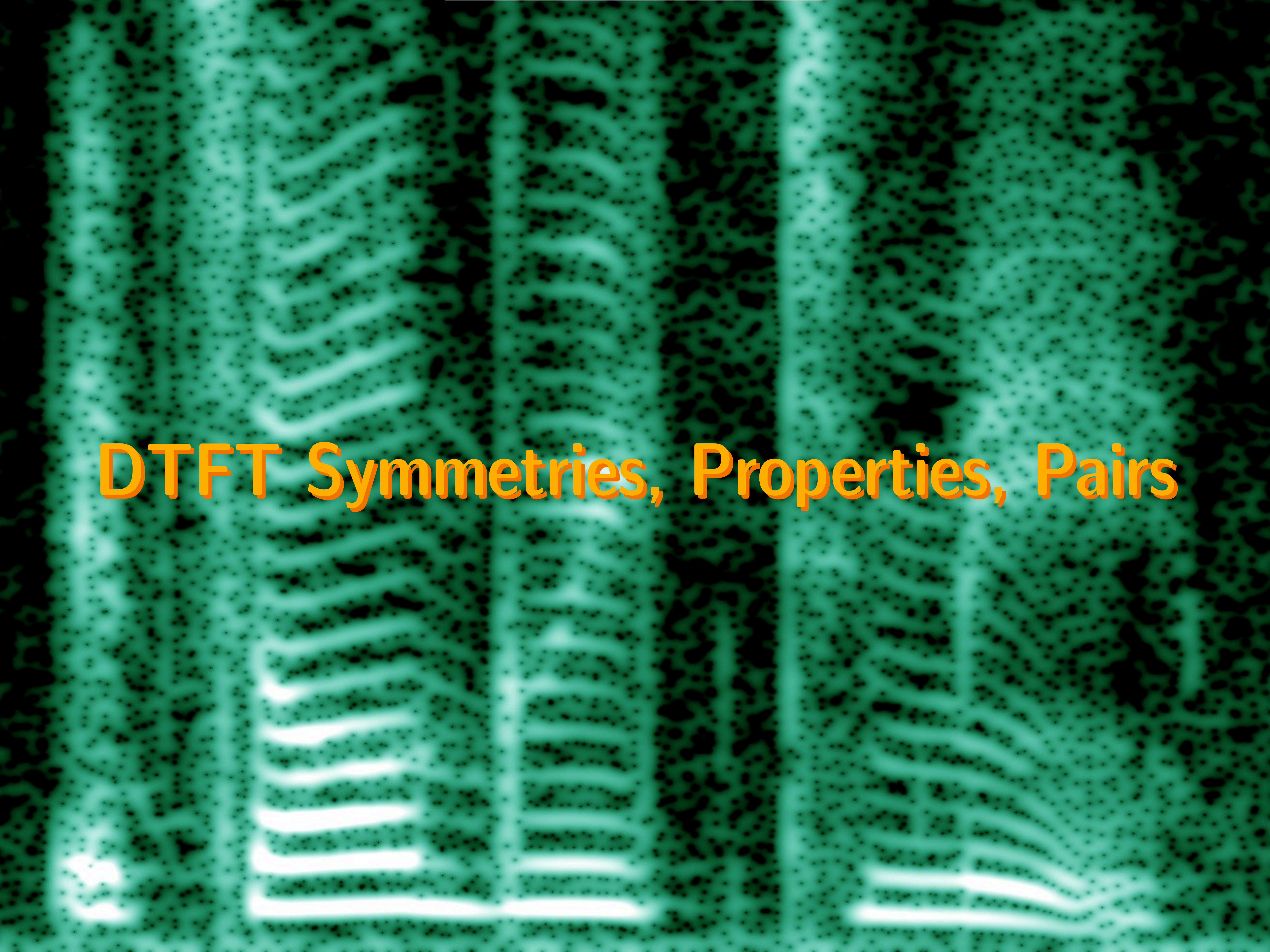
LTI System Responses

Frequency Response	Impulse Response
$H(f) = \|H(f)\|e^{j\angle H(f)}$	$h[n]$

Frequency response $H(f)$ describes how the system modifies a pure frequency $e^{j2\pi fn}$, $-\infty < n < \infty$. The output is given by the product of the input and the frequency response. For general inputs $x[n]$, use DTFT to decompose into linear combinations of pure frequencies

$$x[n] = \int X(f)e^{j2\pi fn}df$$
$$X(f) = \sum x[n]e^{-j2\pi fn}$$

and by LTI properties, the output is $Y(f) = X(f) \cdot H(f)$.



DTFT Symmetries, Properties, Pairs

DTFT Pairs, Properties, Symmetries

Students should be able to understand, derive, and apply:

- **DTFT pairs** in Table 3.1 (page 135)
- **DTFT symmetries** in Table 3.2 & 3.3 (page 142 & 143)
- **DTFT properties** in Table 3.5 (page 189)

Note: Tables and page numbers are for the book "Digital Signal Processing: Principles and Applications" by Thomas Holton, Cambridge, 2021.

DTFT Pairs: Absolutely Summable Sequences

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad \text{(absolutely summable)}$$

$$\delta[n] \longleftrightarrow 1$$

$$\delta[n-k] \longleftrightarrow e^{-j2\pi k}$$

$$\begin{cases} \frac{1}{2N+1}, & -N \le n \le N, \\ 0, & \text{otherwise}, \end{cases} \longleftrightarrow \frac{\sin(\pi f(2N+1))}{(2N+1)\sin(\pi f)}$$

$$\begin{cases} \frac{1}{N+1}, & 0 \le n \le N, \\ 0, & \text{otherwise}, \end{cases} \longleftrightarrow \frac{\sin(\pi f(N+1))}{(N+1)\sin(\pi f)} e^{-j\pi f N}$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j2\pi f}}$$

$$a^n(u[n]-u[n-N]) \longleftrightarrow \frac{1-a^N e^{-j2\pi f N}}{1-ae^{-j2\pi f}}$$

- DTFT sum converges uniformly to a continuous and differentiable function
- Compute DTFT by plugging into analysis formula

DTFT Pairs: Energy Sequences

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \qquad \text{(square summable, finite energy)}$$

$$\frac{\sin(2\pi b n)}{\pi n} \longleftrightarrow \begin{cases} 1, & |f| \leq b \\ 0, & b < |f| \leq \frac{1}{2} \end{cases} \qquad \text{ideal low pass filter}$$

$$\begin{cases} \frac{1}{4}, & n = 0 \\ \frac{(-1)^n - 1}{2\pi^2 n^2}, & n \neq 0 \end{cases} \longleftrightarrow |f|, \quad |f| \leq \frac{1}{2} \qquad \text{proportional gain filter}$$

$$\begin{cases} 0, & n = 0 \\ \frac{1-(-1)^n}{\pi n}, & n \neq 0 \end{cases} \longleftrightarrow \begin{cases} j, & -\frac{1}{2} \leq f < 0 \\ -j, & 0 \leq f \leq \frac{1}{2} \end{cases} \qquad \text{Hilbert transformer}$$

- DTFT sum converges in mean square sense
- DTFT may exhibit discontinuities
- Partial DTFT sum exhibits Gibbs phenomena around discontinuities
- Compute inverse DTFT by plugging into synthesis formula

DTFT Pairs: Power/Periodic Sequences

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty \qquad \text{(finite power)}$$

$$e^{j2\pi f_0 n} \longleftrightarrow \sum_{k=-\infty}^{\infty} \delta(f - f_0 - k)$$

$$\cos(2\pi f_0 n) \longleftrightarrow \frac{1}{2}\left[\sum_k \delta(f - f_0 - k) + \delta(f + f_0 - k)\right]$$

$$\sin(2\pi f_0 n) \longleftrightarrow \frac{1}{2j}\left[\sum_k \delta(f - f_0 - k) - \delta(f + f_0 - k)\right]$$

$$1 \longleftrightarrow \sum_k \delta(f - k)$$

- DTFT sum does not converge for some frequencies, gives rise to Dirac delta functions
- DTFT contains Dirac delta functions
- Compute inverse DTFT by plugging into synthesis formula

DTFT Pairs: Power/Periodic Sequences

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty \qquad \text{(finite power)}$$

$$\sum_{m=-\infty}^{\infty} \delta[n - mN] \quad \longleftrightarrow \quad \frac{1}{N} \sum_{k=0}^{N-1} \sum_{p=-\infty}^{\infty} \delta\left(f - \frac{k}{N} - p\right)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi k n}{N}} \qquad\qquad = \frac{1}{N} \sum_{u=-\infty}^{\infty} \delta\left(f - \frac{u}{N}\right)$$

$$\sum_{m=-\infty}^{\infty} g[n - mN] \quad \longleftrightarrow \quad \frac{1}{N} \sum_{u=-\infty}^{\infty} G\left(\frac{u}{N}\right) \delta\left(f - \frac{u}{N}\right)$$

$$= g[n] * \sum_{m=-\infty}^{\infty} \delta[n - mN]$$

- For periodic signals, expand in Fourier series (DTFS)
- Compute DTFT using DTFT properties such as convolution formula

DTFT Convergence

$$\sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi fn} = H(f)$$

$$\lim_{N\to\infty} \sum_{n=-N}^{N} h[n]e^{-j2\pi fn} = H(f)$$

Given $\varepsilon > 0$, there exists N such that

$$\left| H(f) - \sum_{n=-k}^{k} h[n]e^{-j2\pi fn} \right| < \varepsilon, \quad \text{for all } k \geq N.$$

$$\left| \sum_{n:|n|>k} h[n]e^{-j2\pi fn} \right| < \varepsilon, \text{for all } k \geq N$$

Not much to accumulate in the tails of $h[n]$.

This is point wise convergence for each f.

DTFT Convergence

Suppose $h[n]$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| = L < \infty$$

$$\lim_{N\to\infty} \sum_{n=-N}^{N} |h[n]| = L$$

Given $\varepsilon > 0$, there exists N such that

$$\left| L - \sum_{n=-k}^{k} |h[n]| \right| < \varepsilon, \text{for all } k \geq N.$$

$$\sum_{n:|n|>k} |h[n]| < \varepsilon, \text{for all } k \geq N$$

$$\left| \sum_{n:|n|>k} h[n]e^{-j2\pi fn} \right| \leq \sum_{n:|n|>k} |h[n]| < \varepsilon, \text{for all } k \geq N$$

Absolute summability is a sufficient condition for DTFT convergence.

DTFT Convergence

$$\sum_{n=-\infty}^{\infty} |h[n]|^2 = L < \infty$$

finite energy implies convergence in the mean-square sense:

$$\lim_{N\to\infty} \int_0^1 \left| H(f) - \sum_{n=-N}^{N} h[n] e^{-j2\pi f n} \right|^2 df = 0$$

the energy in the error goes to zero

Computing real and imaginary parts

$$z = a + jb$$

$$z = z^*$$

$$a + jb = a - jb$$

$$a = a$$

$$b = -b = 0$$

one complex equation same as two real equations

Therefore $z = z^* \;\Rightarrow\; z$ is real.

Similarly, $z = -z^* \;\Rightarrow\; z$ is pure imaginary.

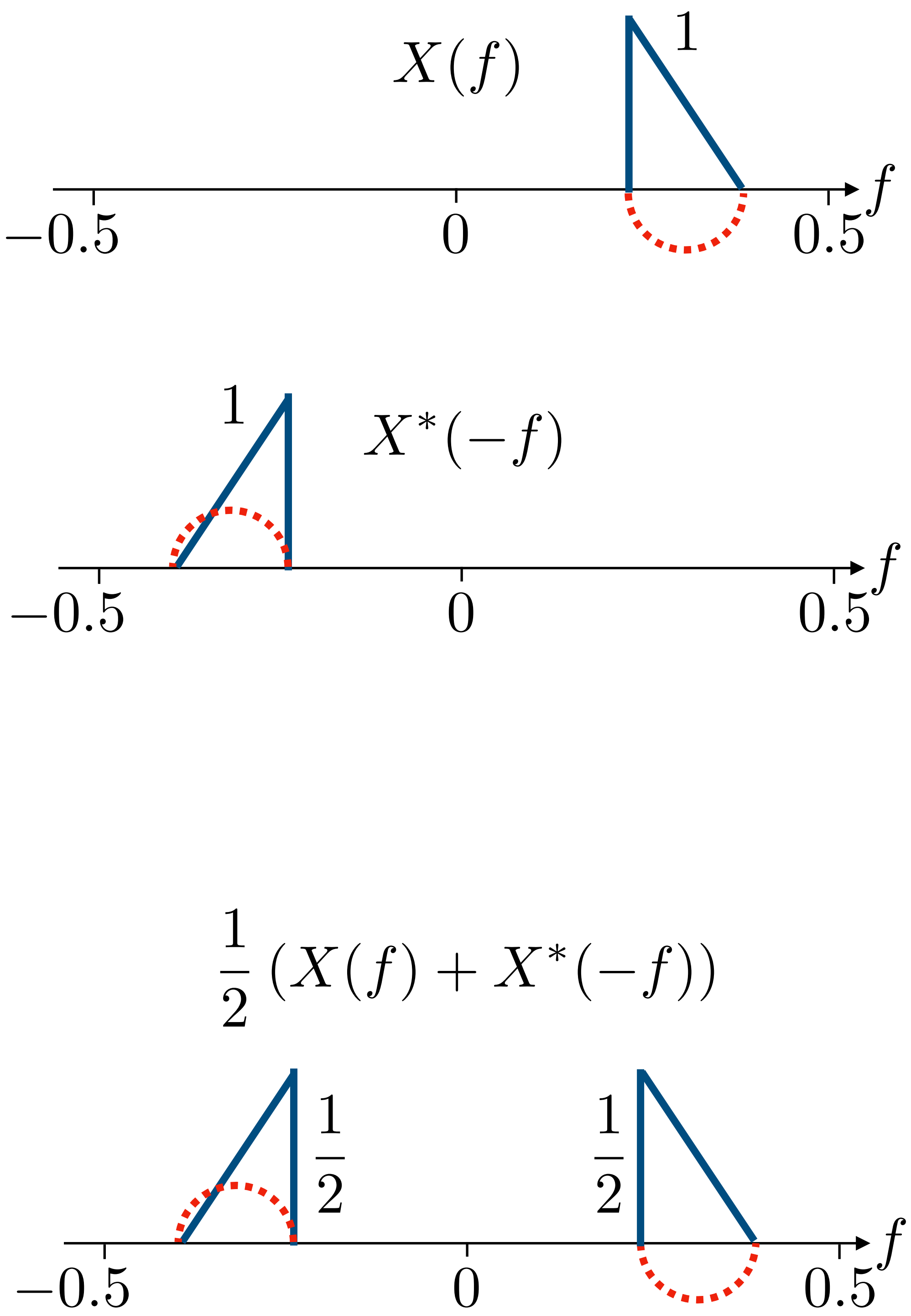
Computing real and imaginary parts

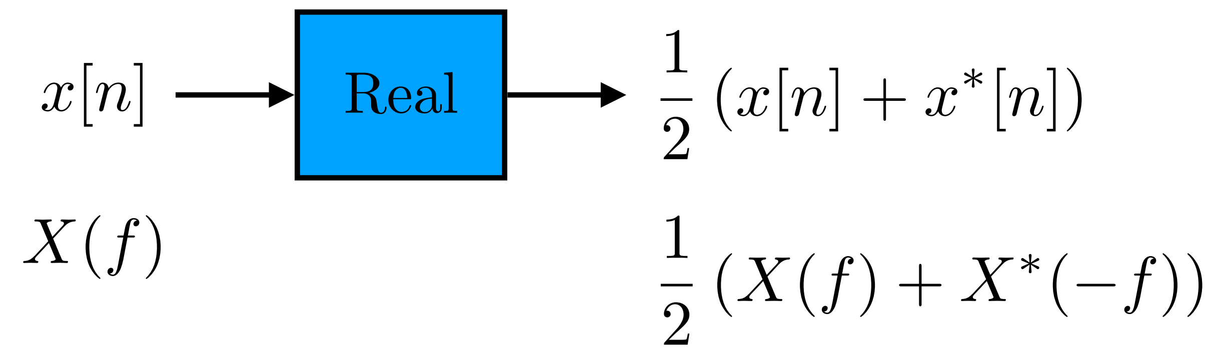
$$z = a + jb$$

$$\frac{1}{2}(z + z^*) = \frac{1}{2}(a + jb + a - jb) = a$$

$$\frac{1}{j2}(z - z^*) = \frac{1}{2}(a + jb - a + jb) = b$$

DTFT Symmetry Example





Taking the real part in the time domain of a complex signal produces the Hermitian part in the frequency domain.

Reversing this process uses the Hilbert transform.

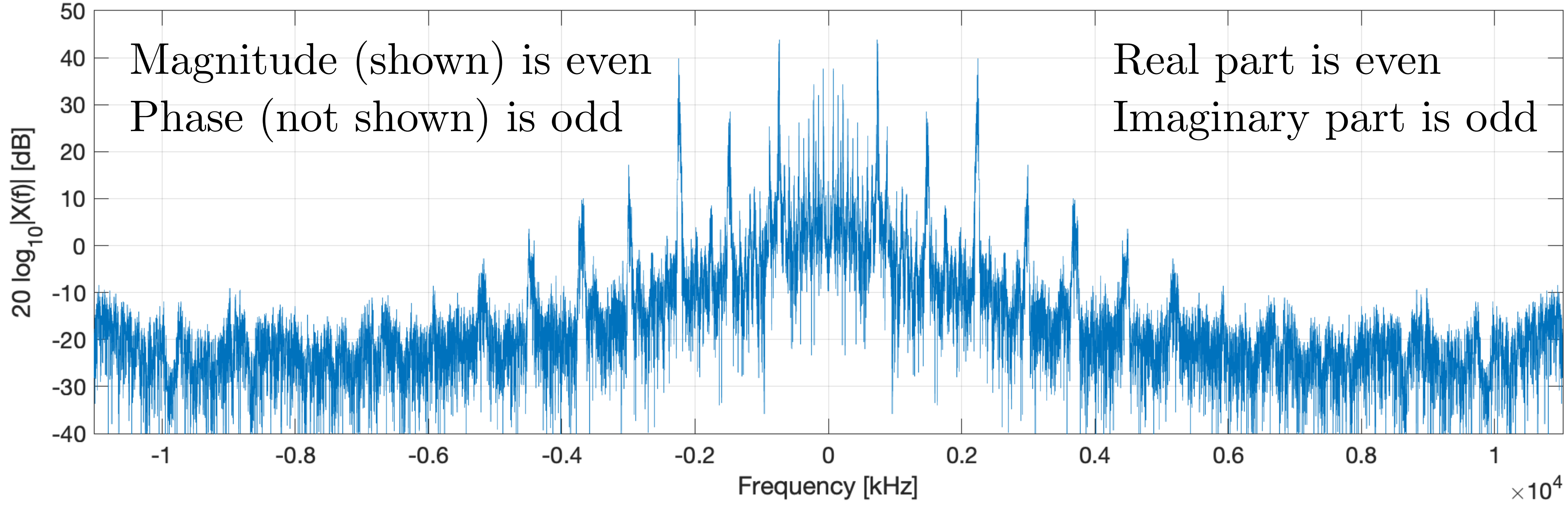
DTFT Symmetry Properties

real	$x[n] = x^*[n]$	$\longleftrightarrow$	$X(f) = X^*(-f)$	Hermitian
Hermitian	$x[n] = x^*[-n]$	$\longleftrightarrow$	$X(f) = X^*(f)$	real
imaginary	$x[n] = -x^*[n]$	$\longleftrightarrow$	$X(f) = -X^*(-f)$	Anti-Hermitian
Anti-Hermitian	$x[n] = -x^*[-n]$	$\longleftrightarrow$	$X(f) = -X^*(f)$	imaginary
even	$x[n] = x[-n]$	$\longleftrightarrow$	$X(f) = X(-f)$	even
odd	$x[n] = -x[-n]$	$\longleftrightarrow$	$X(f) = -X(-f)$	odd

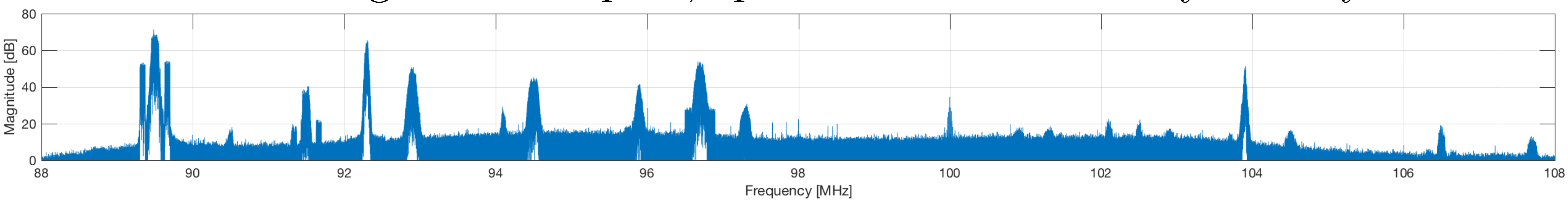
$$\underset{\text{Hermitian}}{X(f) = X^*(-f)} \iff \begin{cases} X_r(f) = X_r(-f) & \text{real part is even} \\ X_i(f) = -X_i(-f) & \text{imaginary part is odd} \end{cases}$$

$$\underset{\text{Hermitian}}{X(f) = X^*(-f)} \iff \begin{cases} |X(f)| = |X(-f)| & \text{magnitude is even} \\ \angle X(f) = -\angle X(-f) & \text{phase is odd} \end{cases}$$

Music signal is real, spectrum exhibits Hermitian symmetry



Radio signal is complex, spectrum exhibits no symmetry



This is the FM band (88-108 MHz) in Logan, Utah.

DTFT Symmetry Example

Two-Way Decompositions of a Sequence

even and odd parts	$$\begin{bmatrix} x_e[n] \\ x_o[n] \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x[n] \\ x[-n] \end{bmatrix}$$
real and imaginary parts	$$\begin{bmatrix} x_r[n] \\ jx_i[n] \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x[n] \\ x^*[n] \end{bmatrix}$$
Hermitian and Anti-Hermitian parts	$$\begin{bmatrix} x_h[n] \\ jx_a[n] \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x[n] \\ x^*[-n] \end{bmatrix}$$

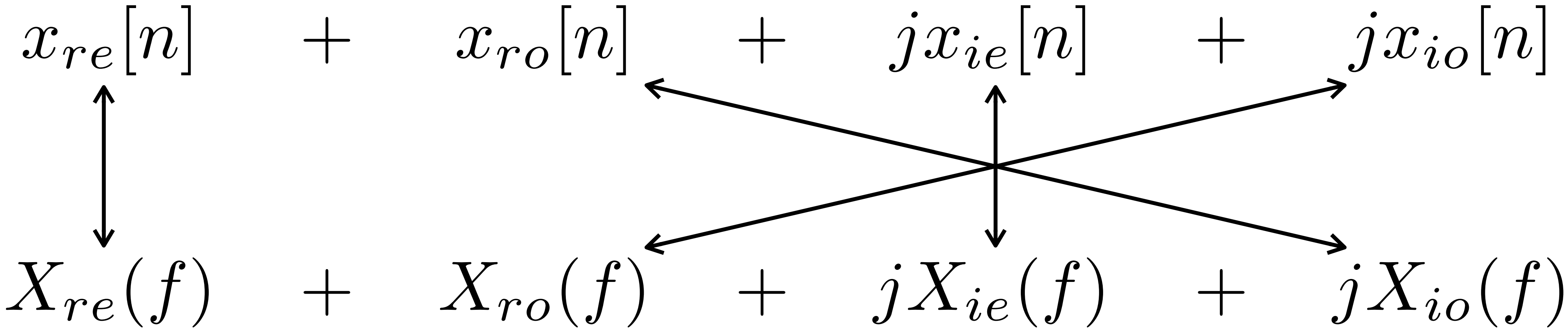
Four-Way Decomposition of a Sequence

$$\begin{bmatrix} x_{re}[n] \\ x_{ro}[n] \\ jx_{ie}[n] \\ jx_{io}[n] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x[n] \\ x[-n] \\ x^*[n] \\ x^*[-n] \end{bmatrix}$$

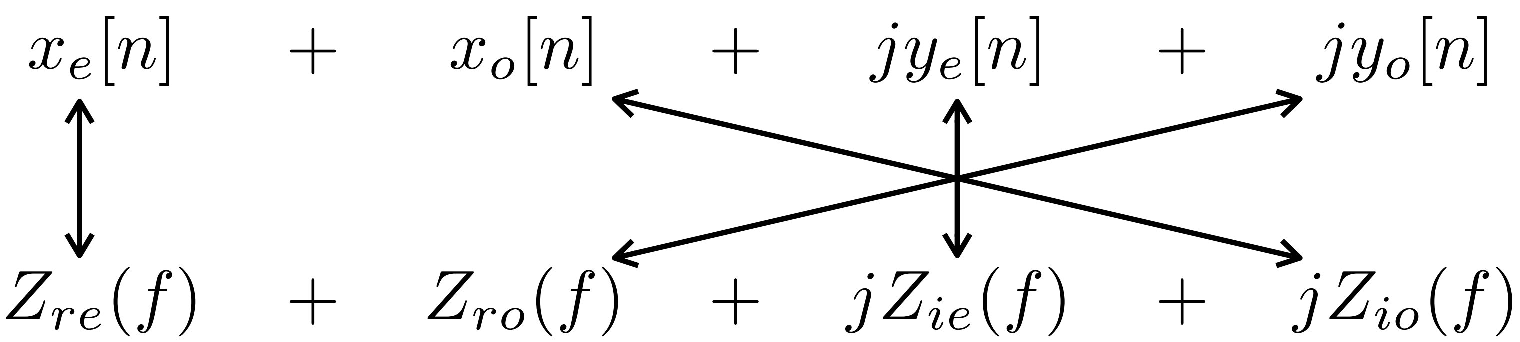
Can apply four-way decomposition to $X(f)$ too.

DTFT Symmetry Example

DTFT transforms four parts independently



Given two real signals $x[n] = x_e[n] + x_o[n]$ and $y[n] = y_e[n] + y_o[n]$, pack $z[n] = x[n] + jy[n]$ and transform to $Z(f)$. The transforms $X(f)$ and $Y(f)$ can be extracted by unpacking the four-way decomposition.



$$X(f) = Z_{re}(f) + jZ_{io}(f)$$
$$jY(f) = Z_{ro}(f) + jZ_{ie}(f)$$

(2 for price of 1)

DTFT Symmetry Example

Suppose the impulse response $h[n]$ is real valued.
What is the output for input $\cos(2\pi f_0 n + \varphi)$, $-\infty < n < \infty$.

$$x[n] = \cos(2\pi f_0 n + \varphi) = \frac{1}{2} e^{j(2\pi f_0 n + \varphi)} + \frac{1}{2} e^{-j(2\pi f_0 n + \varphi)}$$

By linearity we have

$$\begin{aligned} y[n] &= \frac{1}{2} H(f_0) e^{j(2\pi f_0 n + \varphi)} + \frac{1}{2} H(-f_0) e^{-j(2\pi f_0 n + \varphi)} \\ &= \frac{1}{2} |H(f_0)| e^{j(2\pi f_0 n + \varphi + \angle H(f_0))} + \frac{1}{2} |H(-f_0)| e^{j(-2\pi f_0 n - \varphi + \angle H(-f_0))} \end{aligned}$$

Because $h[n]$ is real $H(f_0) = H^*(-f_0)$. Then

$$\begin{aligned} |H(-f_0)| &= |H(f_0)| \\ \angle H(-f_0) &= -\angle H(f_0) \end{aligned}$$

$$\begin{aligned} y[n] &= |H(f_0)| \frac{1}{2} \left(e^{j(2\pi f_0 n + \varphi + \angle H(f_0))} + e^{-j(2\pi f_0 n + \varphi + \angle H(f_0))} \right) \\ &= |H(f_0)| \cos\left(2\pi f_0 n + \varphi + \angle H(f_0)\right) \end{aligned}$$

Everlasting real sinusoids are eigenfunctions of LTI systems with real impulse response.

DTFT Property Example

$$x[n] \longleftrightarrow X(f)$$

$$x[n-d] \longleftrightarrow X(f)e^{-j2\pi f d}$$

Delay in the time domain leads to multiplication by a complex exponential in the frequency domain.

Non-causal zero-phase moving average filter

$$\begin{cases} \frac{1}{2N+1}, & -N \le n \le N, \\ 0, & \text{otherwise}, \end{cases} \longleftrightarrow \frac{\sin(\pi f(2N+1))}{(2N+1)\sin(\pi f)}$$

Causal linear-phase moving average filter

$$\begin{cases} \frac{1}{2N+1}, & 0 \le n \le 2N, \\ 0, & \text{otherwise}, \end{cases} \longleftrightarrow \frac{\sin(\pi f(2N+1))}{(2N+1)\sin(\pi f)}e^{-j2\pi f N}$$

DTFT Property Example

$$x[n] \longleftrightarrow X(f)$$

$$x[n-d] \longleftrightarrow X(f)e^{-j2\pi f d}$$

$$\sum_{k=0}^{N} a[k]y[n-k] = \sum_{m=0}^{M} b[m]x[n-m]$$

$$\left(\sum_{k=0}^{N} a[k]e^{-j2\pi f k}\right) Y(f) = \left(\sum_{k=0}^{M} b[m]e^{-j2\pi f m}\right) X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{B(f)}{A(f)}$$

frequency response of LTI system described by difference equation

$$A(f) = \sum_{k=0}^{N} a[k]e^{-j2\pi f k}$$

$$B(f) = \sum_{m=0}^{M} b[m]e^{-j2\pi f m}$$

DTFT Convolution Property

$$x[n] \longleftrightarrow X(f) \qquad h[n] \longleftrightarrow H(f)$$

$$y[n] = x[n] * h[n] = \sum_k h[k]x[n-k]$$

$$\begin{aligned}
Y(f) &= \sum_n y[n]e^{-j2\pi f n} = \sum_n \sum_k h[k]x[n-k]e^{-j2\pi f n} \\
&= \sum_k h[k] \sum_n x[n-k]e^{-j2\pi f n}, \quad m = n-k, \quad n = m+k \\
&= \left(\sum_k h[k]e^{-j2\pi f k} \right) \cdot \left(\sum_m x[m]e^{-j2\pi f m} \right) \\
&= H(f) \cdot X(f)
\end{aligned}$$

$$x[n] * h[n] \longleftrightarrow X(f) \cdot H(f)$$

DTFT Multiplication Property

$$x[n] \quad \longleftrightarrow \quad X(f) \qquad\qquad h[n] \quad \longleftrightarrow \quad H(f)$$

$$y[n] = x[n] \cdot h[n]$$

$$\begin{aligned}
Y(f) = \sum_n y[n] e^{-j2\pi f n} &= \sum_n h[n] x[n] e^{-j2\pi f n} \\
&= \sum_n h[n] \left(\int_0^1 X(\lambda) e^{j2\pi\lambda n} d\lambda \right) e^{-j2\pi f n} \\
&= \int_0^1 \left(\sum_n h[n] e^{-j2\pi(f-\lambda)n} d\lambda \right) X(\lambda) d\lambda \\
&= \int_0^1 H(f-\lambda) X(\lambda) d\lambda \qquad \text{(periodic convolution)} \\
&= H(f) \circledast X(f)
\end{aligned}$$

$$x[n] \cdot h[n] \quad \longleftrightarrow \quad X(f) \circledast H(f)$$

Important DTFT Properties

Convolution Property

$$x[n] * h[n] \leftrightarrow X(f) \cdot H(f)$$

Filtering (spectral shaping, spectral shaving, etc.)

Multiplication Property

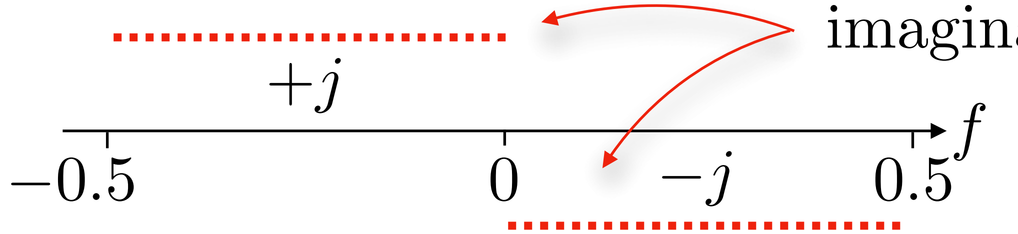
$$x[n] \cdot h[n] \leftrightarrow X(f) \circledast H(f)$$

Windowing and spectral leakage

Mix with sinusoids for spectral shifting (modulation and demodulation)

Mix with impulse train for sampling, upsampling, downsampling, sample rate conversion

Important DTFT Properties: Hilbert Transform



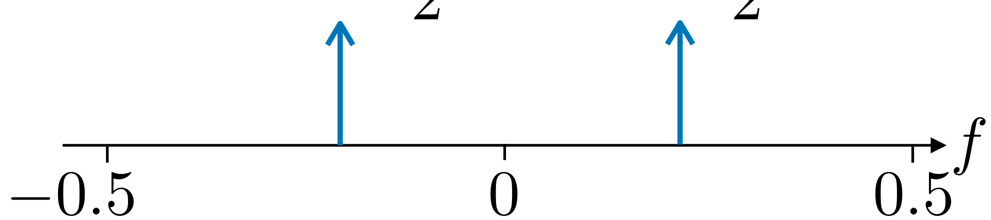
$$H(f) = \begin{cases} +j, & -0.5 \le f < 0 \\ -j, & 0 < f \le 0.5 \end{cases}$$

$$h[n] = \frac{1-(-1)^n}{\pi n} = \cdots, 0, -\frac{2}{3\pi}, 0, -\frac{2}{\pi}, 0, \frac{2}{\pi}, 0, \frac{2}{3\pi}, 0, \cdots$$

$n = 0$

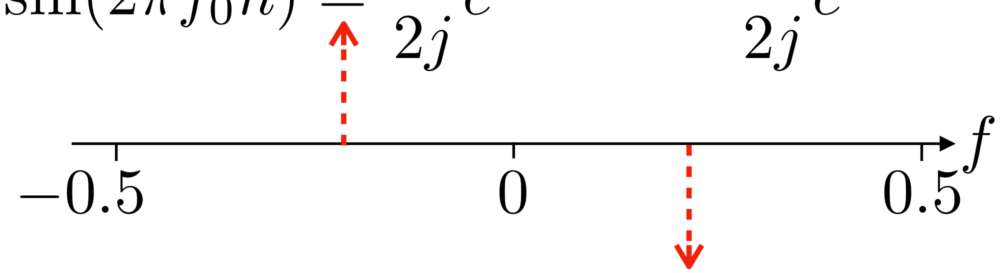
real-odd

$$\cos(2\pi f_0 n) = \frac{1}{2}e^{j2\pi f_0 n} + \frac{1}{2}e^{-j2\pi f n}$$

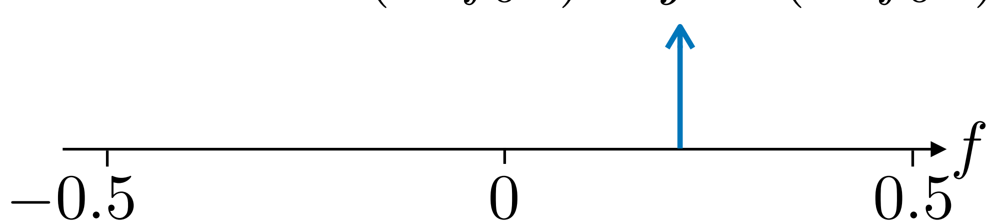


$h[n]$ performs 90 degree phase shift

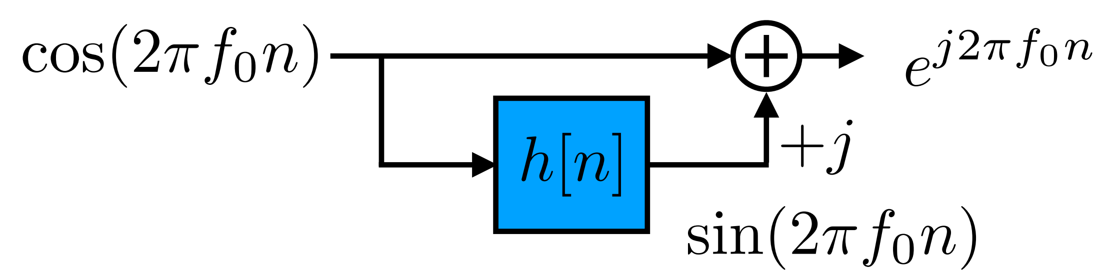
$$\sin(2\pi f_0 n) = \frac{1}{2j}e^{j2\pi f_0 n} - \frac{1}{2j}e^{-j2\pi f n}$$



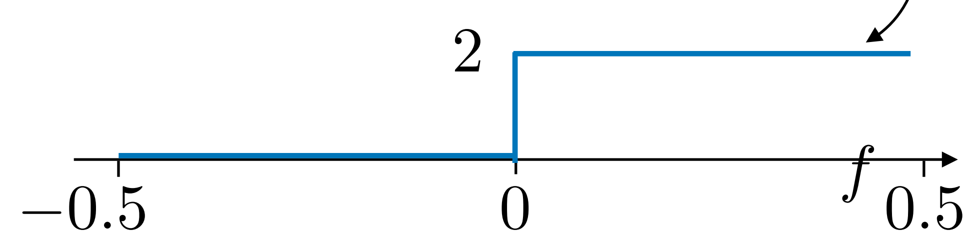
$$e^{j2\pi f_0 n} = \cos(2\pi f_0 n) + j\sin(2\pi f_0 n)$$



Hilbert transform

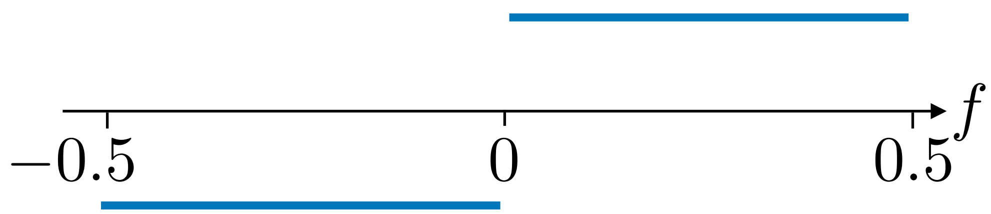


$$\delta[n] + jh[n] \leftrightarrow 1 + jH(f) = 2U(f)$$



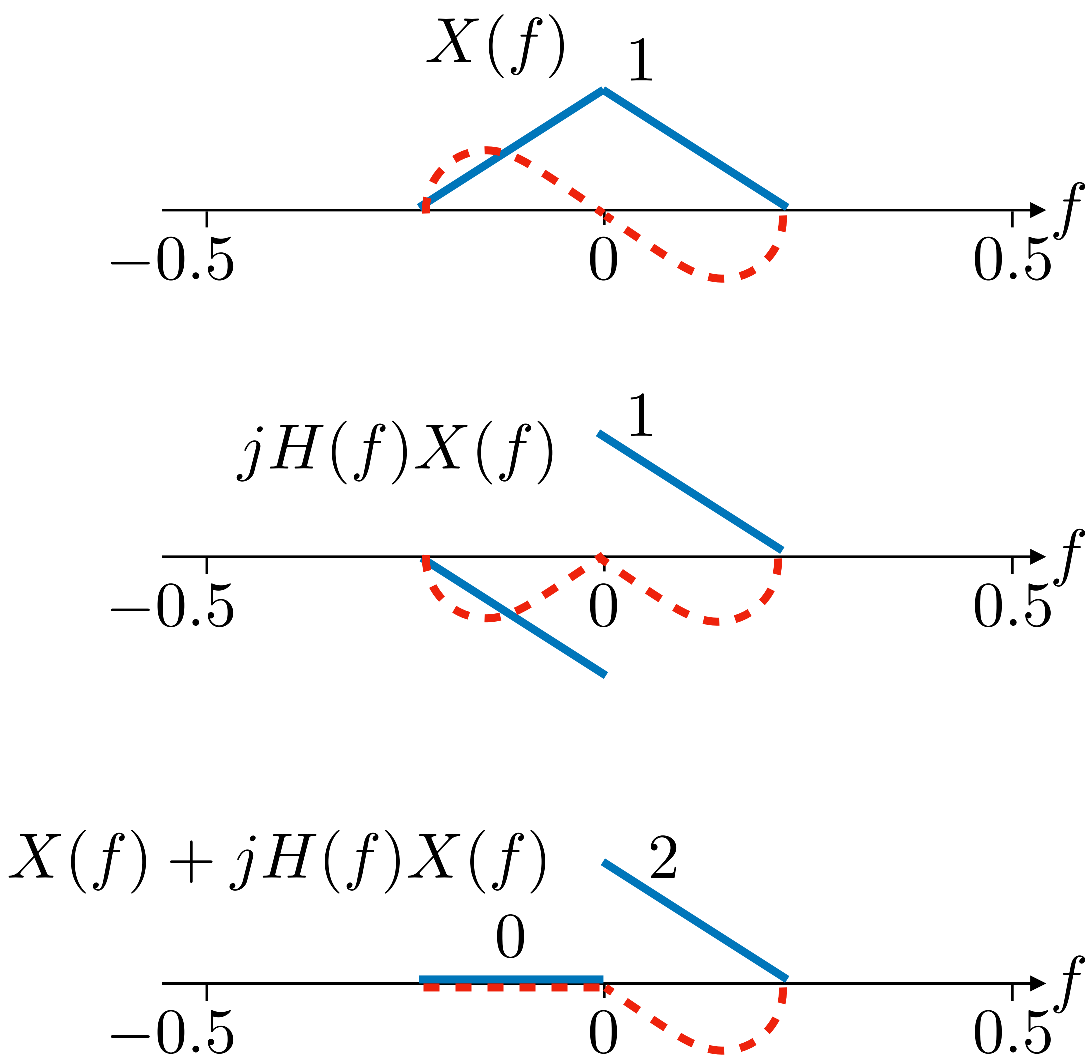
The Hilbert transform removes negative frequencies.

DTFT Properties: Hilbert Transform

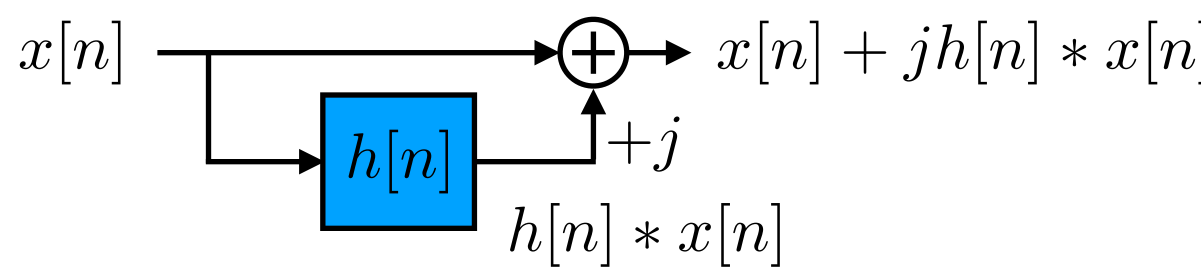


$$jH(f) = \begin{cases} -1, & -0.5 \le f < 0 \\ +1, & 0 < f \le 0.5 \end{cases}$$

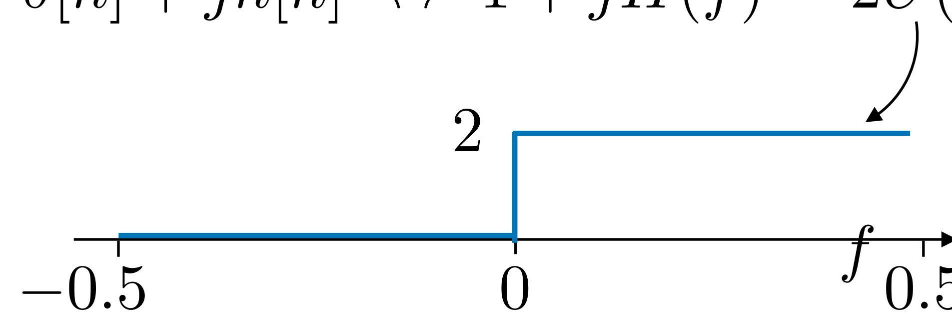
$$x[n] = x^*[n] \ \ \text{(real)} \quad \leftrightarrow \quad X(f) = X^*(-f) \ \ \text{(Hermitian)}$$



Hilbert transform

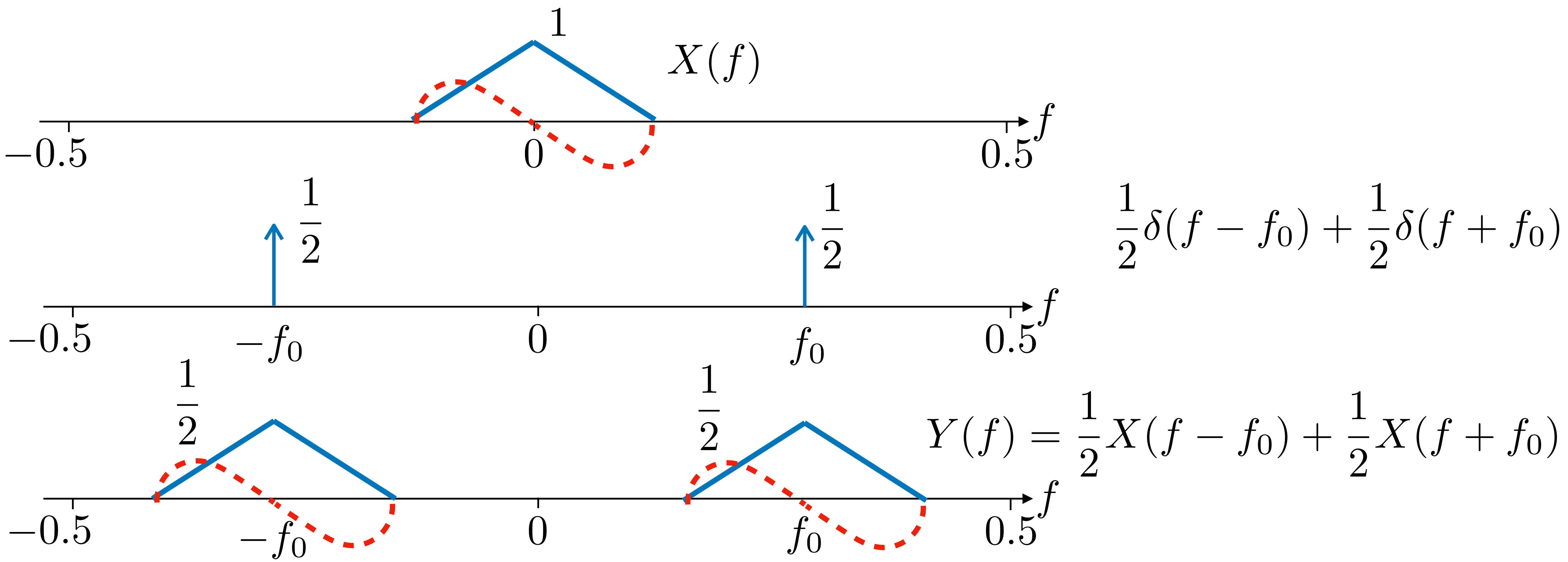


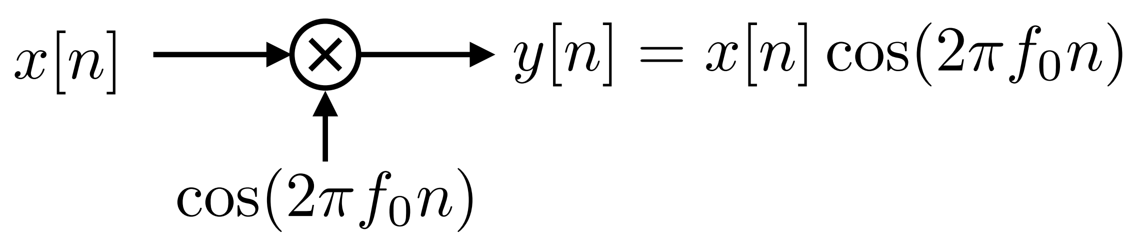
$$\delta[n] + jh[n] \ \leftrightarrow \ 1 + jH(f) = 2U(f)$$



Applications: Single sideband modulation, audio effects, ...

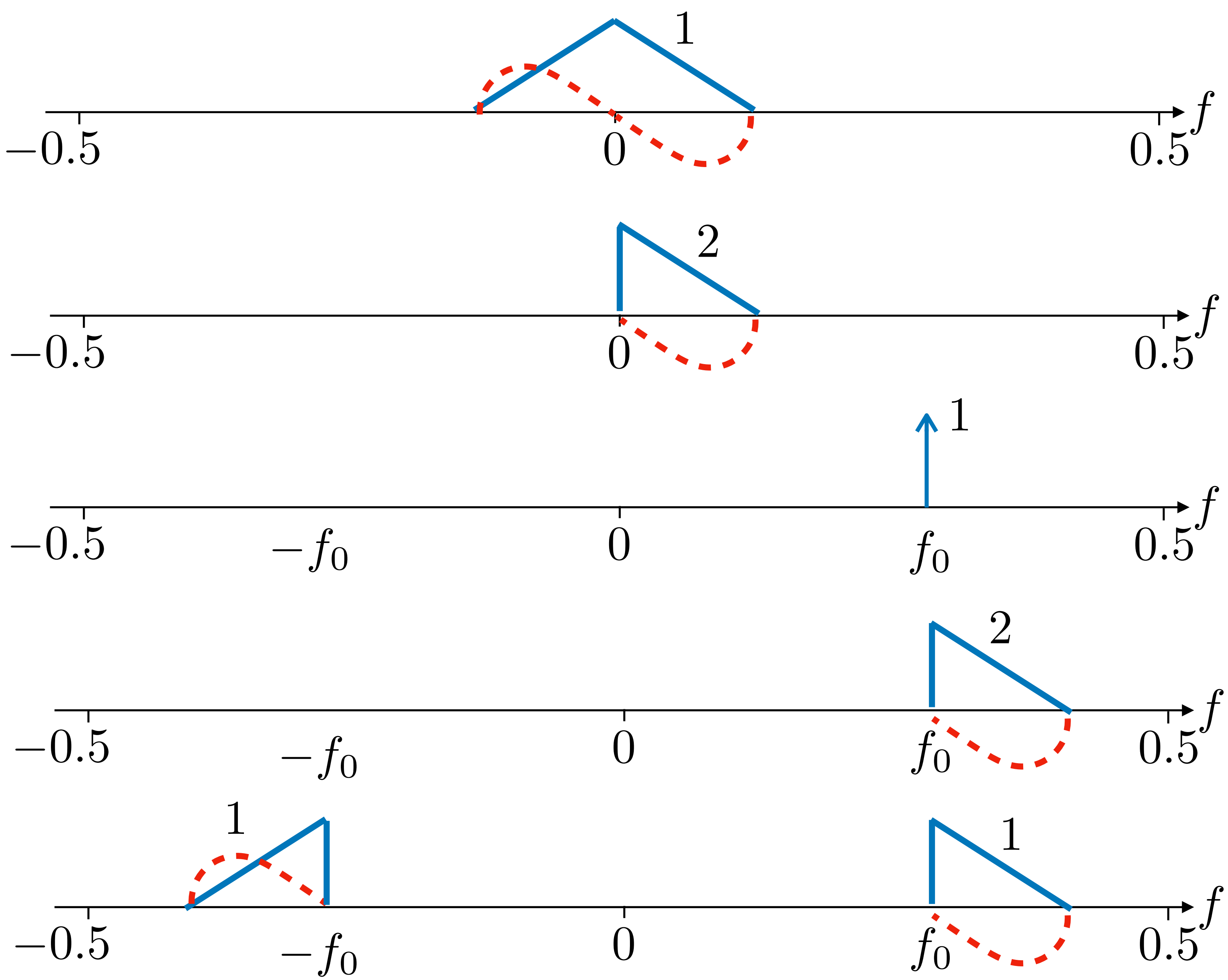
DTFT Properties: Double Side-Band Modulation





Bandwidth of modulated signal is twice
the bandwidth of the baseband signal.

DTFT Properties: Single Side-Band Modulation



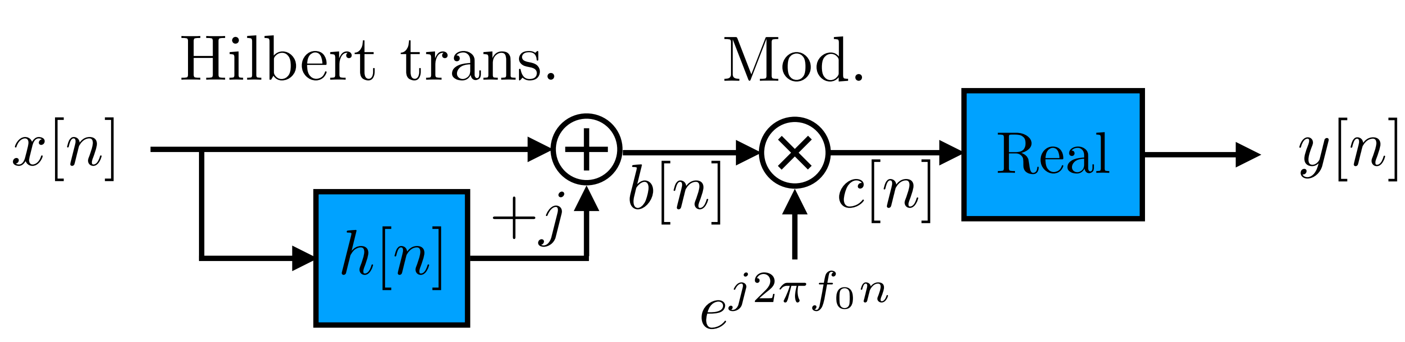
$$X(f)$$

$$B(f) = 2U(f)X(f)$$

$$\delta(f - f_0)$$

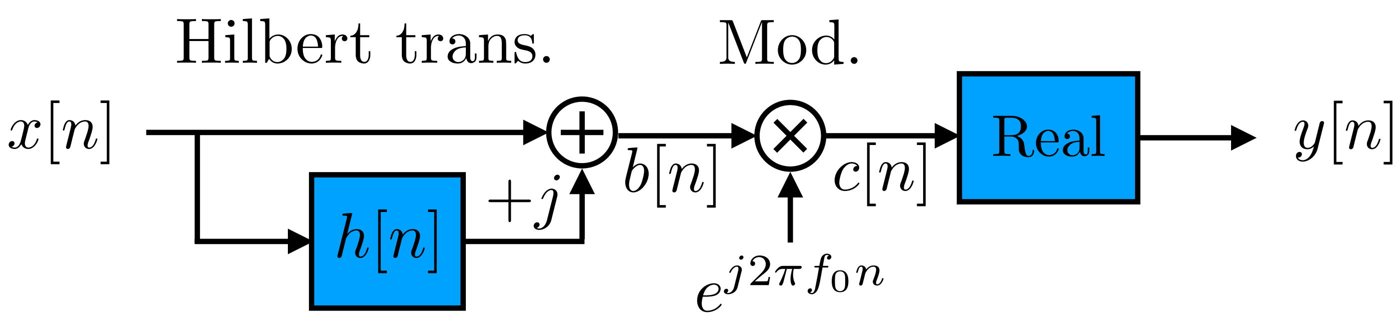
$$C(f) = B(f - f_0)$$

$$Y(f) = \frac{1}{2}\left(C(f) + C^*(-f)\right)$$

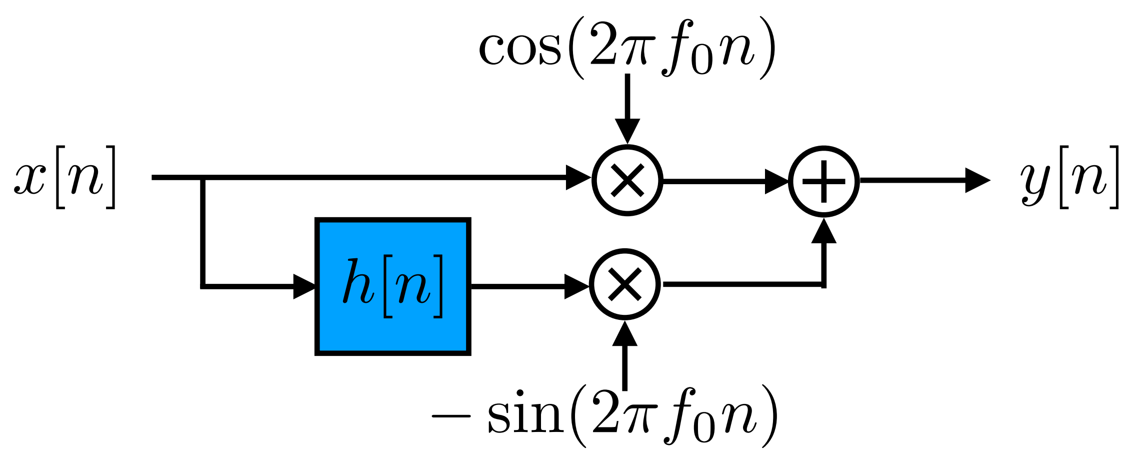


Bandwidth of modulated signal equal to bandwidth of baseband signal.

DTFT Properties: Equivalent Block Diagrams

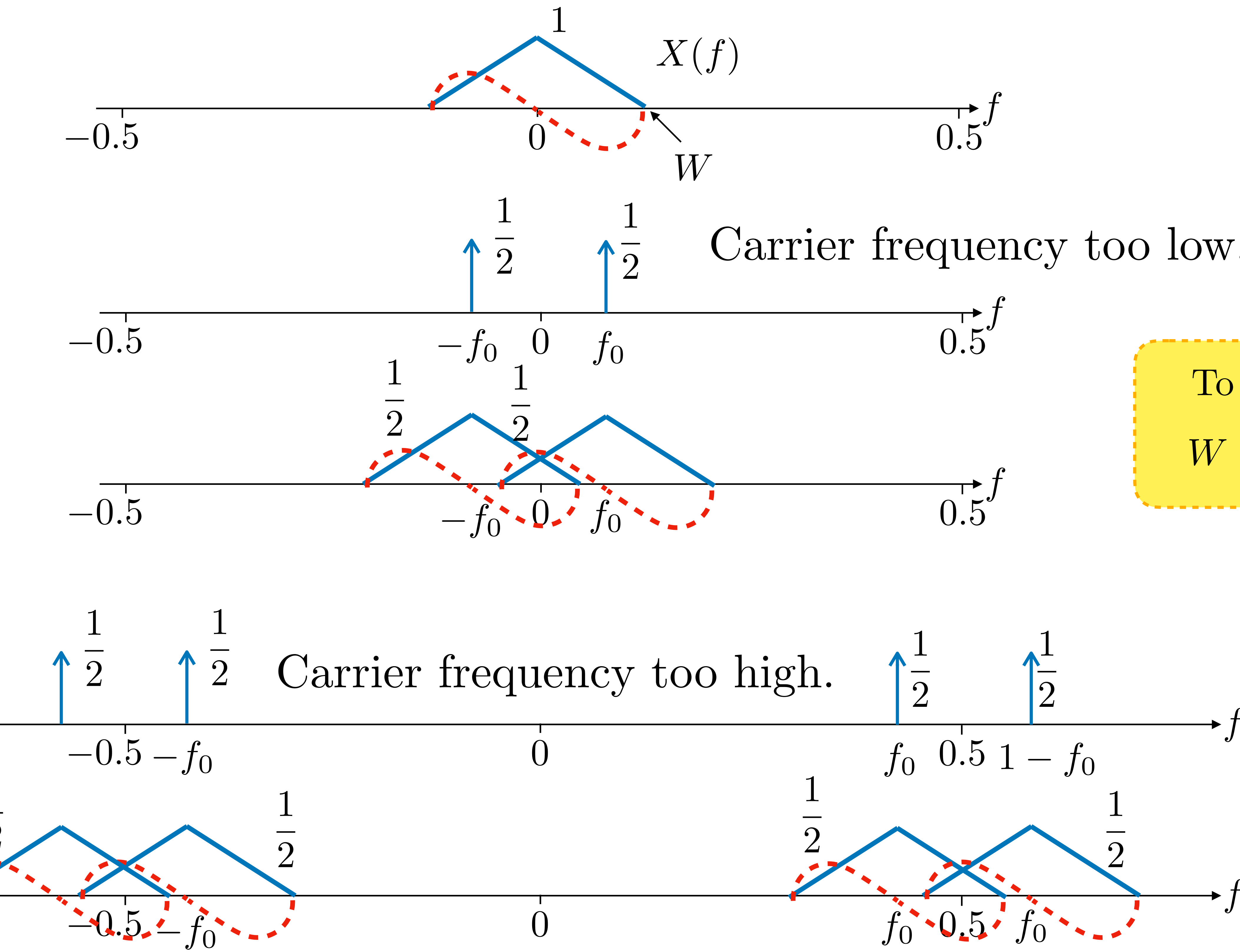


$$\text{Real}\left\{(b_r + jb_i)(c + js)\right\} = b_r \cdot c - b_i \cdot s$$



Work through spectral plots to show equivalence of these two systems.

DTFT Properties: Frequencies that Avoid Overlap



To avoid overlap:

$$W \leq f_0 \leq \frac{1}{2} - W$$

DTFT Properties: More Examples

Convolution and Multiplication Properties
http://classes.ece.usu.edu/3640/resources/DTFT/dtft_properties.html

Shelving Filters for Audio Processing
http://classes.ece.usu.edu/3640/resources/example_shelving_filter.html

Adjustable Bandpass Filter
http://classes.ece.usu.edu/3640/resources/adjustable_bpf/bpftest_diagram.pdf
http://classes.ece.usu.edu/3640/resources/adjustable_bpf/bpftest.pdf
http://classes.ece.usu.edu/3640/resources/adjustable_bpf/rainbowroad.mp3
http://classes.ece.usu.edu/3640/resources/adjustable_bpf/rainbowroad_bpf.mp3

DTFT Properties: Tools

Do this operation in the time domain ...

... to get this effect in the frequency domain.

Spectral Operators/Tools

Time Domain	Frequency Domain
Multiply by $e^{j2\pi f_0 n}$, $\cos(2\pi f_0 n)$, $\sin(2\pi f_0 n)$, or periodic signal.	Shift or replicate
Convolution with impulse response or filter using difference equation	Scale or slice or reshape (multiplication)
Multiplication by window or truncate	Smear/leakage (convolution)

Later we will learn about other operations that stretch and compress in the frequency domain.